Generating functional analysis of Minority Games with real market histories

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Abstract. It is shown how the generating functional method of De Dominicis can be used to solve the dynamics of the original version of the minority game (MG), in which agents observe real as opposed to fake market histories. Here one again finds exact closed equations for correlation and response functions, but now these are defined in terms of two connected effective non-Markovian stochastic processes: a single effective agent equation similar to that of the 'fake' history models, and a second effective equation for the overall market bid itself (the latter is absent in 'fake' history models). The result is an exact theory, from which one can calculate from first principles both the persistent observables in the MG and the distribution of history frequencies.

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1. Introduction

Minority Games (MG) [1] are simple and transparent models which were designed to increase our understanding of the complex collective processes which result from inductive decision making by interacting agents in simplified 'markets'. They are mathematical implementations of the so-called El Farol bar problem [2]. Many versions of the MG have by now been studied in the literature, see e.g. the recent textbook [3] for an overview. They differ in the type of microscopic dynamics used (e.g. batch versus on-line, stochastic versus deterministic), in the definition of the information provided to the agents (real-valued versus discrete, true versus fake market histories) and the agents' decision making strategies, and also in the specific recipe used for converting the observed external information into a trading action (inner products versus look-up tables). Models with 'fake' market histories (proposed first in [4]), where at each point in time all agents are given random rather than real market data upon which to base their decisions, have the advantage of being Markovian and were therefore the first to be studied and solved in the theoretical physics literature using techniques from equilibrium [5, 6, 7, 8] and non-equilibrium [9, 10, 11, 12] statistical mechanics.

After [4] had revealed the similarity between the behaviour of the volatility in the standard MG models with real versus fake market histories, it was shown via numerical

simulations that this statement did not extend to many variations of the MG, such as games with different strategy valuation update rules [13] or with populations where agents do not all observe history strings of the same length [14]. Furthermore, even in the standard MG one does find profound differences in the history frequency distributions (although there these differences do not impact on observables such as the volatility or the fraction of 'frozen' agents). A partly phenomenological attempt at analyzing quantitatively the effects of true history in the MG was presented in [15], and followed by a simulation study [16] of bid periodicities induced by having real histories. After these two papers virtually all theorists restricted themselves to the exclusive analysis of MG versions with fake histories, simply because there is no proper theory yet for MG versions with real histories, in spite of the fact that these are the more realistic types.

There would thus seem to be merit in a mathematical procedure which would allow for the derivation of exact dynamical solutions for MGs with *real* market histories. The objective of this paper is to develop and apply such a procedure. Models with real market histories are strongly non-Markovian, so analytical approaches based on pseudo-equilibrium approximations (which require the existence of a microscopic Lyapunov function) are ruled out. In contrast, the generating functional analysis (GFA) method of [17], which has an excellent track record in solving the dynamics of Markovian MGs, will turn out to work also in the case of non-Markovian models. There are two complications in developing a GFA for MGs with real histories. Firstly, having real histories implies that no 'batch' version of the dynamics can be defined (since batch models by definition involve averaging by hand over all possible histories). Thus one has to return to the original on-line definitions. Secondly, the temporal regularization method [18] upon which one normally relies in carrying out a GFA of on-line MG versions is no longer helpful. This regularization is based on the introduction of random durations of the individual on-line iteration steps of the process, which disrupts the timing of all retarded microscopic forces and thereby leads to extremely messy equations‡. Thus, one has to develop the GFA directly in terms of the un-regularized microscopic laws.

This paper is divided into two distinct parts, similar to the more traditional GFA studies of MGs with fake market histories. The first part deals with the derivation of closed macroscopic laws from which to solve the canonical dynamic order parameters for the standard (on-line) MG with true market history. These will turn out to be formulated in terms of *two* effective equations (rather than a single equation, as for models with fake histories): one for an effective agent, and one for an effective overall market bid. These equations are fully exact. The second part of the paper is devoted to constructing solutions for these effective processes. In particular, this paper focuses on the usual persistent observables of the MG and on the distribution of history frequencies, which are calculated in the form of an expansion of which the first few terms are derived in explicit form. The final results find excellent confirmation in numerical simulations.

‡ Note that in models with fake histories there are no retarded microscopic forces, so that there this particular problem could not occur.

2. Definitions

2.1. Generalized Minority Game with both valuation and overall bid perturbations

In the standard MG one imagines having N agents, labeled by $i = 1, \ldots, N$. At each iteration step $\ell \in \{0, 1, 2, \ldots\}$ of the game, each agent i submits a 'bid' $b_i(\ell) \in \{-1, 1\}$ to the market. The (re-scaled) cumulative market bid at stage ℓ is defined as

$$A(\ell) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} b_i(\ell) + A_e(\ell) \tag{1}$$

An external contribution $A_e(\ell)$ has been added, representing e.g. the actions of market regulators, which will enable us to identify specific response functions later. Profit is assumed to be made by those agents who find themselves subsequently in the minority group, i.e. when $A(\ell) > 0$ by those agents i with $b_i(\ell) < 0$, and when $A(\ell) < 0$ by those with $b_i(\ell) > 0$. Each agent i determines his bid $b_i(\ell)$ at each step ℓ on the basis of publicly available information, which the agents believe to represent historic market data, here given by the vector $\boldsymbol{\lambda}(\ell, A, Z) \in \{-1, 1\}^M$:

$$\boldsymbol{\lambda}(\ell, A, Z) = \begin{pmatrix} \mathrm{sgn}[(1-\zeta)A(\ell-1) + \zeta Z(\ell, 1)] \\ \vdots \\ \mathrm{sgn}[(1-\zeta)A(\ell-M) + \zeta Z(\ell, M)] \end{pmatrix} \tag{2}$$

The numbers $\{Z(\ell, \lambda)\}$, with $\lambda = 1, \ldots, M$, are zero-average Gaussian random variables, which represent a 'fake' alternative to the true market data. M is the number of iteration steps in the past for which market information is made available. We define $\alpha = 2^M/N$, and take α to remain finite as $N \to \infty$. The parameter $\zeta \in [0, 1]$ allows us to interpolate between the cases of strictly true ($\zeta = 0$) and strictly fake ($\zeta = 1$) market histories. We distinguish between two classes of 'fake history' variables:

$$\text{consistent}: \quad Z(\ell, \lambda) = Z(\ell - \lambda), \qquad \langle Z(\ell) Z(\ell') \rangle = \kappa^2 \delta_{\ell\ell'} \tag{3}$$

$$\text{inconsistent}: Z(\ell, \lambda) \text{ all independent}, \quad \langle Z(\ell, \lambda) Z(\ell', \lambda') \rangle = \kappa^2 \delta_{\ell\ell'} \delta_{\lambda\lambda'} \tag{4}$$

We note that (4) does not correspond to a pattern being shifted in time, contrary to what one expects of a string representing the time series of the overall bid, so that the agents in a real market could easily detect that they are being fooled. Hence (3) seems a more natural description of fake history. Although fake, it is at least consistently so.

Each agent has S trading strategies, which we label by $a = 1, \ldots, S$. Each strategy a of each trader i consists of a complete list $\mathbf{R}^{ia}$ of 2^M recommended trading decisions $\{R^{ia}_{\boldsymbol{\lambda}}\} \in \{-1, 1\}$, covering all 2^M possible values of the external information vector $\boldsymbol{\lambda}$. We draw all entries $\{R^{ia}_{\boldsymbol{\lambda}}\}$ randomly and independently before the start of the game, with equal probabilities for ± 1. Upon observing history string $\boldsymbol{\lambda}(\ell, A, Z)$ at stage ℓ, given a trader's active strategy at that stage is $a_i(\ell)$, the agent will follow the instruction of his active strategy and take the decision $b_i(\ell) = R^{ia_i(\ell)}_{\boldsymbol{\lambda}(\ell, A, Z)}$. To determine their active strategies $a_i(\ell)$, all agents keep track of valuations $p_{ia}(\ell)$, which measure how often and

to what extent each strategy a would have led to a minority decision if it had been used from the start of the game onwards. These valuations are updated continually, via

$$p_{ia}(\ell+1) = p_{ia}(\ell) - \frac{\tilde{\eta}}{\sqrt{N}} A(\ell) R^{ia}_{\boldsymbol{\lambda}(\ell,A,Z)} \tag{5}$$

The factor $\tilde{\eta}$ represents a learning rate. If the active strategy $a_i(\ell)$ of trader i at stage ℓ is defined as the one with the highest valuation $p_{ia}(\ell)$ at that point, and upon writing $\mathcal{F}_{\boldsymbol{\lambda}}[\ell, A, Z] = \sqrt{\alpha N}\ \delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ell,A,Z)}$, our process becomes

$$p_{ia}(\ell+1) = p_{ia}(\ell) - \frac{\tilde{\eta}}{N\sqrt{\alpha}} A(\ell) \sum_{\boldsymbol{\lambda}} R^{ia}_{\boldsymbol{\lambda}} \mathcal{F}_{\boldsymbol{\lambda}}[\ell, A, Z] \tag{6}$$

$$A(\ell) = A_e(\ell) + \frac{1}{N\sqrt{\alpha}} \sum_i \sum_{\boldsymbol{\lambda}} R^{ia_i(\ell)}_{\boldsymbol{\lambda}} \mathcal{F}_{\boldsymbol{\lambda}}[\ell, A, Z] \tag{7}$$

$$a_i(\ell) = \arg \max_{a\in\{1,\ldots,S\}} \{p_{ia}(\ell)\} \tag{8}$$

We note that $(\alpha N)^{-1} \sum_{\boldsymbol{\lambda}} 1 = (\alpha N)^{-1} \sum_{\boldsymbol{\lambda}} \mathcal{F}^2_{\boldsymbol{\lambda}}[\ell, A, Z] = 1$. The standard MG is recovered for $\zeta \to 0$ (i.e. true market data only), whereas the 'fake history' MG as in e.g. [4, 9] is found for $\zeta \to 1$ (i.e. fake market data only, of the inconsistent type (4)).

Henceforth we will restrict ourselves to the simplest case $S = 2$, where each agent has only two strategies, so $a \in \{1, 2\}$, since the choice made for S has been shown to have only a quantitative effect on the behaviour of the MG. Our equations can now be simplified in the standard way upon introducing the new variables

$$q_i(\ell) = \frac{1}{2}[p_{i1}(\ell) - p_{i2}(\ell)] \tag{9}$$

$$\boldsymbol{\omega}^i = \frac{1}{2}[\mathbf{R}^{i1} + \mathbf{R}^{i2}], \qquad \boldsymbol{\xi}^i = \frac{1}{2}[\mathbf{R}^{i1} - \mathbf{R}^{i2}] \tag{10}$$

and $\boldsymbol{\Omega} = N^{-1/2} \sum_i \boldsymbol{\omega}^i$. The bid of agent i at step ℓ is now seen to follow from

$$\begin{aligned}\mathbf{R}^{ia_i(\ell)} &= \frac{1}{2}[\mathbf{R}^{i1} - \mathbf{R}^{i2}] + \frac{1}{2}\mathrm{sgn}[q_i(\ell)][\mathbf{R}^{i1} + \mathbf{R}^{i2}] \\ &= \boldsymbol{\omega}^i + \mathrm{sgn}[q_i(\ell)]\boldsymbol{\xi}^i \end{aligned}\tag{11}$$

The above $S = 2$ formulation is easily generalized to include decision noise: one simply replaces $\mathrm{sgn}[q_i(\ell)] \to \sigma[q_i(\ell), z_i(\ell)]$, in which the $\{z_j(\ell)\}$ are independent and zero average random numbers, described by a symmetric and unit-variance distribution $P(z)$. The function $\sigma[q, z]$ is taken to be non-decreasing in q for any z, and parametrized by a control parameter $T \geq 0$ such that $\sigma[q, z] \in \{-1, 1\}$, with $\lim_{T\to 0} \sigma[q, z] = \mathrm{sgn}[q]$ and $\lim_{T\to\infty} \int dz\ P(z)\sigma[q, z] = 0$. Typical examples are additive and multiplicative noise definitions, described by $\sigma[q, z] = \mathrm{sgn}[q + Tz]$ and $\sigma[q, z] = \mathrm{sgn}[q]\ \mathrm{sgn}[1 + Tz]$, respectively. The parameter T measures the degree of randomness in the agents' decision making, with $T = 0$ bringing us back to $a_i(\ell) = \arg\ \max_a\{p_{ia}(\ell)\}$, and with purely random strategy selection for $T = \infty$.

Upon translating our microscopic laws (6,7) into the language of the valuation differences (9) for $S = 2$, we find that now our MG equations close in terms of our new dynamical variables $\{q_i(\ell)\}$, so that perturbations of valuations (again for the

purpose of defining response functions later) can be implemented simply by replacing $q_i(\ell) \to q_i(\ell) + \theta_i(\ell)$, with $\theta_i(\ell) \in \mathbb{R}$. Thus we arrive at the following closed equations, defining our generalized $S = 2$ MG process:

$$q_i(\ell+1) = q_i(\ell) + \theta_i(\ell) - \frac{\tilde{\eta}}{N\sqrt{\alpha}} \sum_{\boldsymbol{\lambda}} \xi^i_{\boldsymbol{\lambda}} \mathcal{F}_{\boldsymbol{\lambda}}[\ell, A, Z] A(\ell) \tag{12}$$

$$A(\ell) = A_e(\ell) + \frac{1}{\sqrt{\alpha N}} \sum_{\boldsymbol{\lambda}} \Big\{ \Omega_{\boldsymbol{\lambda}} + \frac{1}{\sqrt{N}} \sum_i \sigma[q_i(\ell), z_i(\ell)] \xi^i_{\boldsymbol{\lambda}} \Big\} \mathcal{F}_{\boldsymbol{\lambda}}[\ell, A, Z] \tag{13}$$

$$\mathcal{F}_{\boldsymbol{\lambda}}[\ell, A, Z] = \sqrt{\alpha N}\; \delta_{\boldsymbol{\lambda}, \boldsymbol{\lambda}(\ell, A, Z)} \tag{14}$$

$$\boldsymbol{\lambda}(\ell, A, Z) = \begin{pmatrix} \mathrm{sgn}[(1-\zeta)A(\ell-1) + \zeta Z(\ell, 1)] \\ \vdots \\ \mathrm{sgn}[(1-\zeta)A(\ell-M) + \zeta Z(\ell, M)] \end{pmatrix} \tag{15}$$

The values of $\{A(\ell), Z(\ell)\}$ for $\ell \leq 0$ and of the $q_i(0)$ play the role of initial conditions.

The key differences at the mathematical level between MG models with fake history and those with true history as defined above, are in the dependence of the microscopic laws on the past via the history string $\{A(\ell-1), \ldots, A(\ell-M)\}$ occurring in $\boldsymbol{\lambda}(\ell, A, Z) \in \{-1, 1\}^M$, in combination with the presence and role of the zero-average Gaussian random variables $\{Z(\ell, \lambda)\}$.

2.2. Mathematical consequences of having real history

In all generating functional analyses of MGs which have been published so far, the choice $\zeta = 1$ eliminated with one stroke of the pen the dependence of the process on the history $\{A(\ell-1), \ldots, A(\ell-M)\}$. The variables $\{Z(\ell, 1), \ldots, Z(\ell, M)\}$ could subsequently be replaced simply by integer numbers μ, labeling each of the $2^M = p = \alpha N$ possible 'pseudo-histories' that could have been drawn at any given time step ℓ. Here this is no longer possible. The variables $\{Z(\ell, \lambda)\}$ now play the role of random disturbances of the true market history as perceived by the agents, and there is no reason why all possible histories should occur (let alone with equal frequencies) or why some entries $\{Z(\ell, \lambda)\}$ (e.g. those with small values of λ, which corrupt the most recent past in the history string) could not be more important than others. The problem has become *qualitatively* different. One can thus anticipate various mathematical consequences for the generating functional analysis of introducing history into the MG. An early appreciation of these will help us to proceed with the calculation more efficiently.

Firstly, we will have to analyze the original on-line version of the MG; the batch version can no longer exist by definition, since it involves averaging by hand over all possible 'histories' at each iteration step. However, the temporal regularization method of [18] which was employed successfully for the on-line MG with fake history [10], based on introducing Poissonnian distributed real-valued random durations for the individual iterations in (12,13), can in practice no longer be used in the non-Markovian case. The reason for this is the problem which prompted the authors of [10] to add the external perturbations $\theta_i(\ell)$ to the regularized on-line process rather than to the original

equations: whereas in a Markov chain the introduction of random durations for the individual iteration steps only implies a harmless uncertainty in where we are on the time axis, in a system with retarded interactions one would generate very messy equations. We must therefore proceed with our process as it is, without temporal regularization (although we will be able to recover the previous theory in the limit $\zeta \to 1$, as it should). It will in fact turn out that the more direct application of the generating functional method presented in this paper brings the benefit of greater transparency. For instance, the continuity assumptions underlying our use of saddle-point arguments in path integrals become much more clear than they were in [10]. As always we continue to concentrate on the evaluation and disorder averaging of the generating functional

$$Z[\boldsymbol{\psi}] = \langle e^{i\sum_{\ell>0}\sum_i \psi_i(\ell)\sigma[q_i(\ell),z_i(\ell)]}\rangle \tag{16}$$

The brackets in (16) denote averaging over the stochastic process (12,13), whose randomness is here caused by the decision noise $\{\mathbf{z}(\ell)\}$ and the fake history variables $\{Z(\ell,\lambda)\}$. Although (16) looks like the corresponding expressions for batch MGs, here we have to allow for $\ell = \mathcal{O}(N)$. Studying the un-regularized process also implies that one has to be more careful with finite size corrections. This has consequences in working out the disorder average of the generating functional: in previous MG versions one needed only the first two moments of the distribution of the strategy look-up table entries. Here, although one must still expect only the first two moments to play a role in the final theory, the need to keep track initially of the finite size correction terms implies that our equations simplify considerably if, instead of binary strategy entries, we choose the variables $\{R^{ia}_{\boldsymbol{\lambda}}\}$ to be zero-average and unit-variance Gaussian variables.

It will turn out that in our analysis of (16) an important role will be played by the following quantity:

$$\begin{aligned}\overline{W}[\ell,\ell';A,Z] &= \frac{1}{\alpha N}\sum_{\boldsymbol{\lambda}}\mathcal{F}[\ell,A,Z]\mathcal{F}[\ell',A,Z]\\ &= \delta_{\boldsymbol{\lambda}(\ell,A,Z),\boldsymbol{\lambda}(\ell',A,Z)}\end{aligned} \tag{17}$$

This object is a function of the paths $\{A\}$ and $\{Z\}$, and indicates whether or not the histories *as perceived by the agents* at times ℓ and ℓ' are identical (irrespective of the extent to which these 'histories' are true). Its statistics are trivial in the absence of history, but will here generally contain information regarding the recurrence of overall bid trajectories. For reasons of economy we will formulate our theory in terms of the quantity (17), rather than substitute $\delta_{\boldsymbol{\lambda}(\ell,A,Z),\boldsymbol{\lambda}(\ell',A,Z)}$ directly. This will prevent unnecessary future repetition, since it will allow for most of the theory to be applied also to MG models with inner product rather than look-up table definitons for the agents' history-to-action conversion [19].

3. The disorder averaged generating functional

3.1. Evaluation of the disorder average

Rather than first writing the microscopic process in probabilistic form, as in [10], we will express the generating functional (16) as an integral over all possible joint paths of the state vector $\mathbf{q}$ and of the overall bid A, and insert appropriate δ-distributions to enforce the microscopic dynamical equations (12,13), i.e.

$$1 = \prod_{i\ell}\int\left[\frac{d\hat{q}_i(\ell)}{2\pi}\right] e^{i\hat{q}_i(\ell)[q_i(\ell+1)-q_i(\ell)-\theta_i(\ell)+\frac{\tilde{\eta}}{N\sqrt{\alpha}}\sum_{\boldsymbol{\lambda}}\xi^i_{\boldsymbol{\lambda}}\mathcal{F}_{\boldsymbol{\lambda}}[\ell,A,Z]]A(\ell)}$$

$$1 = \prod_{\ell}\int\left[\frac{d\hat{A}(\ell)}{2\pi}\right] e^{i\hat{A}(\ell)[A(\ell)-A_e(\ell)-\frac{1}{\sqrt{\alpha N}}\sum_{\boldsymbol{\lambda}}\left\{\Omega_{\boldsymbol{\lambda}}+\frac{1}{\sqrt{N}}\sum_i\sigma[q_i(\ell),z_i(\ell)]\xi^i_{\boldsymbol{\lambda}}\right\}\mathcal{F}_{\boldsymbol{\lambda}}[\ell,A,Z]}$$

(since our microscopic laws are of an iterative and causal form, they have unique solutions). To compactify our equations we will use the short-hand $s_i(\ell) = \sigma[q_i(\ell), z_i(\ell)]$. We can now write the disorder average $\overline{Z[\boldsymbol{\psi}]}$ of (16) as follows:

$$\begin{aligned}\overline{Z[\boldsymbol{\psi}]} = &\int\left[\prod_{\ell>0}\frac{dA(\ell)d\hat{A}(\ell)}{2\pi}e^{i\hat{A}(\ell)[A(\ell)-A_e(\ell)]}\right]\\ &\times\Bigg\langle\int\left[\prod_{i\ell}\frac{dq_i(\ell)d\hat{q}_i(\ell)}{2\pi}e^{i\hat{q}_i(\ell)[q_i(\ell+1)-q_i(\ell)-\theta_i(\ell)]+i\psi_i(\ell)s_i(\ell)}\right]\\ &\overline{\times\ e^{\frac{i}{N\sqrt{\alpha}}\sum_{\boldsymbol{\lambda}}\sum_{i\ell}\left[\tilde{\eta}\hat{q}_i(\ell)\xi^i_{\boldsymbol{\lambda}}A(\ell)-\hat{A}(\ell)\left(\omega^i_{\boldsymbol{\lambda}}+s_i(\ell)\xi^i_{\boldsymbol{\lambda}}\right)\right]\mathcal{F}_{\boldsymbol{\lambda}}[\ell,A,Z]}}\Bigg\rangle_{\{\mathbf{z},Z\}}\end{aligned} \tag{18}$$

The brackets $\langle\ldots\rangle_{\{\mathbf{z},Z\}}$ denote averaging over the Gaussian decision noise and the pseudo-memory variables, and we have used the abbreviations (10). The short-hand $Du = (2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}u^2}$ and the previously introduced quantity $\overline{W}[\ldots]$ in (17) allow us to write the disorder average (over the independently distributed zero-average and unit-variance $R^{ia}_{\boldsymbol{\lambda}}$) in the last line of (18) as

$$\begin{aligned}\overline{e^{\frac{i}{N\sqrt{\alpha}}\sum_{\boldsymbol{\lambda}}\sum_{i\ell}\cdots}} &= \prod_{\boldsymbol{\lambda}}\prod_i\int Du\ e^{\frac{iu}{2N\sqrt{\alpha}}\sum_\ell\left[\tilde{\eta}\hat{q}_i(\ell)A(\ell)-\hat{A}(\ell)[1+s_i(\ell)]\right]\mathcal{F}_{\boldsymbol{\lambda}}[\ell,A,Z]}\\ &\quad\times\prod_{\boldsymbol{\lambda}}\prod_i\int Dv\ e^{\frac{iv}{2N\sqrt{\alpha}}\sum_\ell\left[\tilde{\eta}\hat{q}_i(\ell)A(\ell)+\hat{A}(\ell)[1-s_i(\ell)]\right]\mathcal{F}_{\boldsymbol{\lambda}}[\ell,A,Z]}\\ &= e^{-\frac{1}{4N}\sum_{\ell\ell'>0}\overline{W}[\ell,\ell';A,Z]\sum_i\left[\tilde{\eta}\hat{q}_i(\ell)A(\ell)-\hat{A}(\ell)s_i(\ell)\right]\left[\tilde{\eta}\hat{q}_i(\ell')A(\ell')-\hat{A}(\ell')s_i(\ell')\right]}\\ &\quad\times\ e^{-\frac{1}{4}\sum_{\ell\ell'>0}\hat{A}(\ell)\overline{W}[\ell,\ell';A,Z]\hat{A}(\ell')}\end{aligned} \tag{19}$$

We next isolate the usual observables $L(\ell,\ell') = N^{-1}\sum_i\hat{q}_i(\ell)\hat{q}_i(\ell')$, $K(\ell,\ell') = N^{-1}\sum_i s_i(\ell)\hat{q}_i(\ell')$, and $C(\ell,\ell') = N^{-1}\sum_i s_i(\ell)s_i(\ell')$, by inserting appropriate integrals over δ-distributions. We also use the abbreviations $\mathcal{D}C = \prod_{\ell\ell'}[\sqrt{N}dC(\ell,\ell')/\sqrt{2\pi}]$ (similarly for other two-time observables) and $\mathcal{D}A = \prod_{\ell>0}[dA(\ell)/\sqrt{2\pi}]$ (similarly for $\hat{A}$). Initial conditions for the $q_i(0)$ are assumed to be of the factorized form $p_0(\mathbf{q}) = \prod_i p_0(q_i(0))$. In anticipation of issues to arise in subsequent stages of our analysis, especially those related to the scaling with N of the number of individual

iterations of the process, we will also define the largest iteration step in the generating functional as $\ell_{\max}$. All this allows us to write $\overline{Z[\boldsymbol{\psi}]}$ in the form

$$
\begin{aligned}
\overline{Z[\boldsymbol{\psi}]} = & \int \mathcal{D}C\mathcal{D}\hat{C}\mathcal{D}K\mathcal{D}\hat{K}\mathcal{D}L\mathcal{D}\hat{L}\ e^{iN\sum_{\ell\ell'}[\hat{C}(\ell,\ell')C(\ell,\ell')+\hat{K}(\ell,\ell')K(\ell,\ell')+\hat{L}(\ell,\ell')L(\ell,\ell')]} \\
& \times\ e^{\mathcal{O}(\ell_{\max}^2\log N)} \int \mathcal{D}A\mathcal{D}\hat{A}\ e^{i\sum_\ell \hat{A}(\ell)[A(\ell)-A_e(\ell)]} \\
& \times\ e^{\frac{1}{4}\tilde{\eta}\sum_{\ell\ell'}\overline{W}[\ell,\ell';A,Z]\left\{\hat{A}(\ell)K(\ell,\ell')A(\ell')+\hat{A}(\ell')K(\ell',\ell)A(\ell)\right\}} \\
& \times\ e^{-\frac{1}{4}\sum_{\ell\ell'}\overline{W}[\ell,\ell';A,Z]\left\{\tilde{\eta}^2A(\ell)L(\ell,\ell')A(\ell')+\hat{A}(\ell)[1+C(\ell,\ell')]\hat{A}(\ell')\right\}} \\
& \times \left\langle \int \prod_{i\ell}\left[\frac{dq_i(\ell)d\hat{q}_i(\ell)}{2\pi}e^{i\hat{q}_i(\ell)[q_i(\ell+1)-q_i(\ell)-\theta_i(\ell)]+i\psi_i(\ell)s_i(\ell)}\right]\cdot\prod_i p_0(q_i(0)) \right. \\
& \qquad \left. \times \prod_i e^{-i\sum_{\ell\ell'}\{\hat{L}(\ell,\ell')\hat{q}_i(\ell)\hat{q}_i(\ell')+\hat{K}(\ell,\ell')s_i(\ell)\hat{q}_i(\ell')+\hat{C}(\ell,\ell')s_i(\ell)s_i(\ell')\}} \right\rangle_{\{\mathbf{z},Z\}} \\
= & \int \mathcal{D}C\mathcal{D}\hat{C}\mathcal{D}K\mathcal{D}\hat{K}\mathcal{D}L\mathcal{D}\hat{L}\ e^{N[\Psi+\Omega+\Phi]+\mathcal{O}(\ell_{\max}^2\log N)}
\end{aligned}
\tag{20}
$$

with

$$
\Psi = i\sum_{\ell\ell'\leq\ell_{\max}}[\hat{C}(\ell,\ell')C(\ell,\ell')+\hat{K}(\ell,\ell')K(\ell,\ell')+\hat{L}(\ell,\ell')L(\ell,\ell')] \tag{21}
$$

$$
\begin{aligned}
\Phi = \frac{1}{N}\log\Big\langle \int & \mathcal{D}A\mathcal{D}\hat{A}\ e^{i\sum_{\ell\leq\ell_{\max}}\hat{A}(\ell)[A(\ell)-A_e(\ell)]} \\
& \times\ e^{-\frac{1}{4}\sum_{\ell\ell'\leq\ell_{\max}}\overline{W}[\ell,\ell';A,Z]M[\ell,\ell';A,\hat{A}]}\Big\rangle_{\{Z\}}
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
\Omega = & \frac{1}{N}\sum_i\log\Big\langle \int \left[\prod_{\ell=0}^{\ell_{\max}}\frac{dq(\ell)d\hat{q}(\ell)}{2\pi}\right]p_0(q(0)) \\
& \times\ e^{i\sum_{\ell\leq\ell_{\max}}\left[\hat{q}(\ell)[q(\ell+1)-q(\ell)-\theta_i(\ell)]+\psi_i(\ell)\sigma[q(\ell),z(\ell)]\right]-i\sum_{\ell\ell'\leq\ell_{\max}}\hat{q}(\ell)\hat{L}(\ell,\ell')\hat{q}(\ell')} \\
& \times\ e^{-i\sum_{\ell\ell'\leq\ell_{\max}}\left[\hat{C}(\ell,\ell')\sigma[q(\ell),z(\ell)]\sigma[q(\ell'),z(\ell')]+\hat{K}(\ell,\ell')\sigma[q(\ell),z(\ell)]\hat{q}(\ell')\right]}\Big\rangle_{\mathbf{z}}
\end{aligned}
\tag{23}
$$

and with

$$
\begin{aligned}
M[\ell,\ell';A,\hat{A}] = & \tilde{\eta}^2A(\ell)L(\ell,\ell')A(\ell')-\tilde{\eta}[\hat{A}(\ell)K(\ell,\ell')A(\ell')+\hat{A}(\ell')K(\ell',\ell)A(\ell)] \\
& +\hat{A}(\ell)[1+C(\ell,\ell')]\hat{A}(\ell')
\end{aligned}
\tag{24}
$$

The $\mathcal{O}(\ell_{\max}^2\log N)$ corrections in (20) are constants, which reflect the scaling with N used in defining the conjugate order parameters.

Compared to the Markovian (fake history) MG versions, we note that Ψ and Ω take their conventional forms, and that all the complications induced by having true market history are concentrated in the function $\Phi[C,K,L]$, which is now defined in terms of a stochastic process for the overall bid $A(\ell)$ rather than being an explicit function of the order parameters (which had been the situation in all fake history versions of the game), and in the remaining task to implement an appropriate scaling with N of the time scale $\ell_{\max}$. We can now also see the advantage in our earlier decision to define Gaussian rather than binary look-up table entries. With the N-scaling of $\ell_{\max}$ still pending, instead of

(19), in the binary case we would have found

$$\overline{e^{\frac{i}{N\sqrt{\alpha}}\sum_{\boldsymbol{\lambda}}\sum_{i\ell}\cdots}} = e^{\sum_i\sum_{\boldsymbol{\lambda}}\log\cos\left[\frac{1}{2N\sqrt{\alpha}}\sum_{\ell\leq\ell_{\max}}\left[\tilde{\eta}\hat{q}_i(\ell)A(\ell)-\hat{A}(\ell)[1+s_i(\ell)]\right]\mathcal{F}_{\boldsymbol{\lambda}}[\ell,A,Z]\right]}$$
$$\times\ e^{\sum_i\sum_{\boldsymbol{\lambda}}\log\cos\left[\frac{1}{2N\sqrt{\alpha}}\sum_{\ell\leq\ell_{\max}}\left[\tilde{\eta}\hat{q}_i(\ell)A(\ell)+\hat{A}(\ell)[1-s_i(\ell)]\right]\mathcal{F}_{\boldsymbol{\lambda}}[\ell,A,Z]\right]} \tag{25}$$

In this expression we see that, for $\ell_{\max} = \mathcal{O}(N)$, the different choices of strategy look-up table entry distribution will give the same results only for those paths $\{A, Z\}$ where the frequency of occurrence each of the 2^M possible histories is of order $\mathcal{O}(N^{-1})$. In the latter case the function $\mathcal{F}_{\boldsymbol{\lambda}}[\ell, A, Z]$ scales effectively inside summations over ℓ as $\mathcal{F}_{\boldsymbol{\lambda}}[\ell, A, Z] = \mathcal{O}(N^{-\frac{1}{2}})$, and we return to (19). Thus, for non-Gaussian distributions of the $\{R^{ia}_{\boldsymbol{\lambda}}\}$ at this stage of the GFA one either has to carry on with the more complicated expression (25), which cannot be expressed in terms of the order parameters $\{C, K, L\}$, or one has to make further assumptions on the overall bid statistics, which (although turning out to be correct) require validation *a posteriori*.

3.2. Canonical time scaling

For the on-line MG with random external information (i.e. with $\zeta = 1$) it is known that the relevant time scale is $\ell_{\max} = \mathcal{O}(N)$. Rather than imposing the time scale $\ell_{\max} = \mathcal{O}(N)$ by hand, it is satisfactory to see that one can also extract this canonical time scaling from our present equations (20,21,22,23).

For finite $\ell_{\max}$ we immediately find $\lim_{N\to\infty}\Phi = 0$ in (22), and our generating functional will be dominated by the physical saddle-point of $\lim_{N\to\infty}[\Psi + \Omega]$, giving $\hat{C} = \hat{K} = \hat{L} = 0$. This leads to a trivial effective single spin problem, which just describes a frozen state. This makes perfect sense in view of our definitions (12,13): individual updates of the variables q_i are of order $N^{-\frac{1}{2}}$, so nothing can change on time-scales corresponding to only a finite number of iteration steps. Thus our present equations automatically lead us to the choice $\ell_{\max} = \mathcal{O}(1/\delta_N)$, where $\lim_{N\to\infty}\delta_N = 0$; the function Φ will indeed scale differently as soon as $\ell_{\max}$ is allowed to diverge with N. We thus define $\ell_{\max} = t_{\max}/\delta_N$, where $0 \leq t_{\max} < \infty$ (of order N^0) and with $\lim_{N\to\infty}\delta_N = 0$. In order to obtain well-defined limits at the end in (21), we see that we have to re-scale our conjugate order parameters according to $(\hat{C}, \hat{K}, \hat{L}) \to {\delta_N}^2(\hat{C}, \hat{K}, \hat{L})$. Furthermore, for the perturbation fields $\{\theta_i, \psi_i\}$ to retain statistical significance they also will have to be re-scaled in the familiar manner, according to $(\theta_i, \psi_i) \to {\delta_N}^{-1}(\tilde{\theta}_i, \tilde{\psi}_i)$ (similar to [10]). The integrations over order parameters and conjugate order parameters in (20) will now become path integrals for $N \to \infty$§.

It will be convenient to introduce the following effective measure:

$$\langle g[\{q,\hat{q},z\}]\rangle_\star = \lim_{N\to\infty}\frac{1}{N}\sum_i \frac{\int\prod_{\ell=1}^{t_{\max}/\delta_N}[dq(\ell)d\hat{q}(\ell)]\langle M_i[\{q,\hat{q},z\}]g[\{q,\hat{q},z\}]\rangle_{\mathbf{z}}}{\int\prod_{\ell=1}^{t_{\max}/\delta_N}[dq(\ell)d\hat{q}(\ell)]\langle M_i[\{q,\hat{q},z\}]\rangle_{\mathbf{z}}} \tag{26}$$

§ This is the point, therefore, where the inevitable continuity assumptions regarding our macroscopic dynamic observables enter. In the present derivation these take a more transparent form than in [10], where they were hidden inside the details of the temporal regularization.

$$
\begin{aligned}
M_i[\{q,\hat{q},z\}] \;&= p_0(q(0))e^{i\delta_N \sum_{\ell=1}^{t_{\max}/\delta_N} \hat{q}(\ell)\left[\frac{q(\ell+1)-q(\ell)}{\delta_N}-\tilde{\theta}_i(\ell)\right]+i\delta_N \sum_\ell \tilde{\psi}_i(\ell)\sigma[q(\ell),z(\ell)]} \\
&\times\; e^{-i\delta_N{}^2\sum_{\ell\ell'=1}^{t_{\max}/\delta_N}\left[\hat{L}(\ell,\ell')\hat{q}(\ell)\hat{q}(\ell')+\hat{K}(\ell,\ell')\sigma[q(\ell),z(\ell)]\hat{q}(\ell')+\hat{C}(\ell,\ell')\sigma[q(\ell),z(\ell)]\sigma[q(\ell'),z(\ell')]\right]}
\end{aligned}
\tag{27}
$$

Upon substituting $\ell_{\max} = t_{\max}/\delta_N$ into our equations (21,22,23), followed by appropriate re-scaling of the conjugate order parameters, these three functions acquire the following form (modulo irrelevant constants):

$$
\Psi = i\delta_N{}^2 \sum_{\ell\ell' \le t_{\max}/\delta_N} \left[\hat{C}(\ell,\ell')C(\ell,\ell') + \hat{K}(\ell,\ell')K(\ell,\ell') + \hat{L}(\ell,\ell')L(\ell,\ell')\right] \tag{28}
$$

$$
\Phi = \frac{1}{N}\log\left\langle \int \mathcal{D}A\mathcal{D}\hat{A}\ \mathcal{W}[A,\hat{A}|Z]\right\rangle_{\{Z\}} \tag{29}
$$

$$
\Omega = \frac{1}{N}\sum_i \log\int \prod_{\ell=1}^{t_{\max}/\delta_N} [dq(\ell)d\hat{q}(\ell)]\ \langle M_i[\{q,\hat{q},z\}]\rangle_{\mathbf{z}} \tag{30}
$$

with

$$
\mathcal{W}[A,\hat{A}|Z] = e^{i\sum_{\ell=1}^{t_{\max}/\delta_N} \hat{A}(\ell)[A(\ell)-A_e(\ell)]-\frac{1}{4}\sum_{\ell\ell'=1}^{t_{\max}/\delta_N}\overline{W}[\ell,\ell';A,Z]M[\ell,\ell';A,\hat{A}]} \tag{31}
$$

It is clear that Ψ and Ω now have proper $N \to \infty$ limits. The canonical choice of δ_N is subsequently determined by the mathematical condition that $\lim_{N\to\infty}\Phi[C,K,L] \neq 0$, but finite. It follows that (20) is again dominated by its physical saddle-point, and we are nearly back in familiar territory.

3.3. The saddle point equations

In order to eliminate the fields $\{\psi_i(\ell),\theta_i(\ell)\}$, and thereby simplify our equations, we next extract the physical meaning of our order parameters from the generating functional by taking appropriate derivatives with respect to these fields. This gives

$$
C(\ell,\ell') = \lim_{N\to\infty}\frac{1}{N}\sum_i \overline{\langle s_i(\ell)s_i(\ell')\rangle} = \langle\sigma[q(\ell),z(\ell)]\sigma[q(\ell'),z(\ell')]\rangle_\star \tag{32}
$$

$$
G(\ell,\ell') = \lim_{N\to\infty}\frac{1}{N}\sum_i \frac{\partial\overline{\langle s_i(\ell)\rangle}}{\partial\theta_i(\ell')} = -i\langle\sigma[q(\ell),z(\ell)]\hat{q}(\ell)\rangle_\star \tag{33}
$$

$$
0 = \lim_{N\to\infty}\frac{1}{N}\sum_i \frac{\partial^2 1}{\partial\theta_i(\ell)\partial\theta_i(\ell')} = -\langle\hat{q}(\ell)\hat{q}(\ell')\rangle_\star \tag{34}
$$

Thus at the physical saddle-point of (20) we have the usual relations $L(\ell,\ell') = 0$ and $K(\ell,\ell') = iG(\ell,\ell')$, where G denotes the single-site response function. Upon varying $\{\hat{C},\hat{K},\hat{L}\}$ in (20) we reproduce self-consistently the by now standard equations

$$
C(\ell,\ell') = \langle\sigma[q(\ell),z(\ell)]\sigma[q(\ell'),z(\ell')]\rangle_\star \tag{35}
$$

$$
G(\ell,\ell') = -i\langle\sigma[q(\ell),z(\ell)]\hat{q}(\ell')\rangle_\star \tag{36}
$$

$$
L(\ell,\ell') = \langle\hat{q}(\ell)\hat{q}(\ell')\rangle_\star = 0 \tag{37}
$$

We turn to variation of the order parameters $\{C,K,L\}$ in $\Psi+\Phi$ (as Ω only depends on the conjugate order parameters). In working out derivatives of Φ we observe that

the conjugate bids effectively act as differential operators, i.e. $\hat{A}(s) \to i\partial/\partial A_e(s)$. This gives us our remaining three saddle point equations:

$$\hat{C}(s,s') = \lim_{N\to\infty} \frac{i}{4N{\delta_N}^2} \frac{\frac{\partial^2}{\partial A_e(s)\partial A_e(s')} \left\langle \int \mathcal{D}A\mathcal{D}\hat{A}\ \mathcal{W}[A,\hat{A}|Z]\ \overline{W}[s,s';A,Z] \right\rangle_{\{Z\}}}{\left\langle \int \mathcal{D}A\mathcal{D}\hat{A}\ \mathcal{W}[A,\hat{A}|Z] \right\rangle_{\{Z\}}} \tag{38}$$

$$\hat{K}(s,s') = \lim_{N\to\infty} \frac{-\tilde{\eta}}{2N{\delta_N}^2} \frac{\frac{\partial}{\partial A_e(s)} \left\langle \int \mathcal{D}A\mathcal{D}\hat{A}\ \mathcal{W}[A,\hat{A}|Z]\ \overline{W}[s,s';A,Z]A(s') \right\rangle_{\{Z\}}}{\left\langle \int \mathcal{D}A\mathcal{D}\hat{A}\ \mathcal{W}[A,\hat{A}|Z] \right\rangle_{\{Z\}}} \tag{39}$$

$$\hat{L}(s,s') = \lim_{N\to\infty} \frac{-i\tilde{\eta}^2}{4N{\delta_N}^2} \frac{\left\langle \int \mathcal{D}A\mathcal{D}\hat{A}\ \mathcal{W}[A,\hat{A}|Z]\ \overline{W}[s,s';A,Z]A(s)A(s') \right\rangle_{\{Z\}}}{\left\langle \int \mathcal{D}A\mathcal{D}\hat{A}\ \mathcal{W}[A,\hat{A}|Z] \right\rangle_{\{Z\}}} \tag{40}$$

At the physical saddle-point, we may use $L = 0$ and the symmetry of $\overline{W}[\ldots]$ to simplify the function $M[\ell,\ell';A,\hat{A}]$ which occurs in the measure (31) to

$$M[\ell,\ell';A,\hat{A}] = \hat{A}(\ell)[1 + C(\ell,\ell')]\hat{A}(\ell') - 2i\tilde{\eta}\hat{A}(\ell)G(\ell,\ell')A(\ell') \tag{41}$$

The generating fields $\{\tilde{\psi}_i(\ell)\}$ are now no longer needed and can be removed. The perturbations $\tilde{\theta}_i$ are still useful for calculating the response function G, but can be chosen site-independent, i.e. $\tilde{\theta}_i(\ell) = \tilde{\theta}(\ell)$. The measure (27) will then lose its site dependence. Also the functions $\{\Psi, \Phi, \Omega\}$ have at this stage become obsolete. We may define a new time $t = \ell\delta_N = \mathcal{O}(N^0)$, which will be real-valued as $N \to \infty$, and we may take the limit $N \to \infty$ in the definitions of our observables. The latter can subsequently be written in terms of the new real-valued time arguments, $C(\ell,\ell') \to C(t,t')$ (and similar for the other kernels).

4. The resulting theory

4.1. Simplification of saddle-point equations

We may now summarize our saddle-point equations for $\{C, G\}$ in the usual compact way, in terms of an effective single agent process:

$$C(t,t') = \langle \mathrm{sgn}[q(t)]\mathrm{sgn}[q(t')]\rangle_\star \qquad G(t,t') = -i\langle \mathrm{sgn}[q(t)]\hat{q}(t')\rangle_\star \tag{42}$$

with a measure which is defined in terms of path integrals, as in [10] (and with time integrals running from $t = 0$ to $t = t_{\max}$):

$$\langle g[\{q,\hat{q},z\}]\rangle_\star = \frac{\int\{dqd\hat{q}\}\ \langle M[\{q,\hat{q},z\}]\ g[\{q,\hat{q},z\}]\rangle_{\mathbf{z}}}{\int\{dqd\hat{q}\}\ \langle M[\{q,\hat{q},z\}]\rangle_{\mathbf{z}}} \tag{43}$$

$$\begin{aligned} M[\{q,\hat{q},z\}] &= p_0(q(0))\ e^{i\int dt\ \hat{q}(t)\left[\frac{d}{dt}q(t) - \theta(t) - \int dt' \hat{K}(t',t)\sigma[q(t'),z(t')]\right]} \\ &\quad \times\ e^{-i\int dt dt' \left[\hat{L}(t,t')\hat{q}(t)\hat{q}(t') + \hat{C}(t,t')\sigma[q(t),z(t)]\sigma[q(t'),z(t')]\right]} \end{aligned} \tag{44}$$

To find the kernels $\{\hat{C}, \hat{K}, \hat{L}\}$ we have to evaluate equations (38,39,40) further, remembering that the left-hand sides as yet still involve the integer time labels (s, s'), rather than the continuous times. Now the scaling chosen for δ_N with N which we adopt will be crucial. We observe that all complications are contained in the evaluation,

for large N and for any given realization of the fake market information path $\{Z\}$, of objects of the following general form (with all operators evaluated at the saddle-point):

$$\langle Q[\{A\}]\rangle_{\{A|Z\}} = \int \mathcal{D}A\mathcal{D}\hat{A}\ \mathcal{W}[A,\hat{A}|Z]\ Q[\{A\}] \tag{45}$$

We can confirm, by repeating the steps taken in evaluating the disorder-averaged generating functional $\overline{Z[\boldsymbol{\psi}]}$ but now for calculating averages of arbitrary functions of the overall market bid path $\{A\}$, that the physical interpretation of the measure (45) is

$$\lim_{N\to\infty} \overline{\langle Q[\{A\}]\rangle} = \Big\langle \langle Q[\{A\}]\rangle_{\{A|Z\}} \Big\rangle_{\{Z\}} \tag{46}$$

Thus (45) defines the asymptotic disorder-averaged probability density for observing a 'path' $\{A\}$ of global bids, for a given realization of the fake history path $\{Z\}$. To evaluate (45) we introduce two path-dependent matrices $G[A,Z]$ and $D[A,Z]$, with entries

$$G[A,Z](\ell,\ell') = \overline{W}[\ell,\ell';A,Z]G(\ell,\ell') \tag{47}$$

$$D[A,Z](\ell,\ell') = \overline{W}[\ell,\ell';A,Z][1+C(\ell,\ell')] \tag{48}$$

Definition (17) tells us that $G[A,Z](\ell,\ell') = G(\ell,\ell')$ if the 'history' observed at stage ℓ is identical to that observed at stage ℓ', and zero otherwise, and similarly for the relation between $D[A,Z](\ell,\ell')$ and $1+C(\ell,\ell')$. We now use auxiliary integration variables $\{\phi_\ell\}$ to linearize the term in the exponent of (45) which is quadratic in $\hat{A}$, and use causality of the response function G where appropriate:

$$\begin{aligned}
\langle Q[\{A\}]\rangle_{\{A|Z\}} &= \int \prod_{\ell=1}^{t_{\max}/\delta_N} \left[\frac{dA(\ell)d\hat{A}(\ell)}{2\pi} e^{i\hat{A}(\ell)[A(\ell)-A_e(\ell)+\frac{1}{2}\tilde{\eta}\sum_{\ell'<\ell} G[A,Z](\ell,\ell')A(\ell')]}\right] Q[\{A\}] \\
&\quad \times \frac{\int [\prod_{\ell=1}^{t_{\max}/\delta_N} d\phi_\ell] e^{-\sum_{\ell\ell'=1}^{t_{\max}/\delta_N} \phi_\ell (D^{-1}[A,Z])_{\ell\ell'}\phi_{\ell'} - i\sum_{\ell=1}^{t_{\max}/\delta_N} \phi_\ell \hat{A}_\ell}}{\int [\prod_\ell d\phi_\ell] e^{-\sum_{\ell\ell'=1}^{t_{\max}/\delta_N} \phi_\ell (D^{-1}[A,Z])_{\ell\ell'}\phi_{\ell'}}} \\
&= \int \Big[\prod_{\ell=1}^{t_{\max}/\delta_N} dA(\ell)\Big]\ Q[\{A\}] \\
&\quad \times \left\langle \prod_{\ell=1}^{t_{\max}/\delta_N} \delta\left[A(\ell) - A_e(\ell) + \frac{1}{2}\tilde{\eta}\sum_{\ell'<\ell} G[A,Z](\ell,\ell')A(\ell') - \phi_\ell\right]\right\rangle_{\{\phi|A,Z\}}
\end{aligned}$$

Here $\langle\ldots\rangle_{\{\phi|A,Z\}}$ refers to averaging over the zero-average Gaussian fields ϕ_ℓ with $\{A,Z\}$-dependent covariance $\langle\phi_\ell\phi_{\ell'}\rangle_{\{\phi|A,Z\}} = \frac{1}{2}D[A,Z](\ell,\ell')$. We conclude from our expression for $\langle Q[\{A\}]\rangle_{\{A|Z\}}$ that the conditional disorder-averaged probability density $\mathcal{P}[\{A\}|\{Z\}]$ for finding a bid path $\{A\}$, given a realization $\{Z\}$ of the pseudo-history, is given by

$$\mathcal{P}[\{A\}|\{Z\}] = \left\langle \prod_{\ell=1}^{t_{\max}/\delta_N} \delta\left[A(\ell) - A_e(\ell) + \frac{1}{2}\tilde{\eta}\sum_{\ell'<\ell} G(\ell,\ell')\overline{W}[\ell,\ell';A,Z]A(\ell') - \phi_\ell\right]\right\rangle_{\{\phi|A,Z\}} \tag{49}$$

with $\langle Q[\{A\}]\rangle_{\{A|Z\}} = \int[\prod_\ell dA(\ell)]\mathcal{P}[\{A\}|\{Z\}]Q[\{A\}]$. Causality ensures that the density (49) is normalized, since both ϕ_ℓ and $G[A,Z](\ell,\ell')$ involve only entries of the paths $\{A,Z\}$ with times $k<\ell$.

Having established (49), our equations (38,39,40) can be simplified considerably. We immediately find that $\hat{C} = 0$. To simplify comparison with the theory of [10] (corresponding to $\zeta = 1$), we will make a final change in notation and put

$$\hat{K}(\ell,\ell') = -\alpha R(\ell',\ell) \qquad \hat{L}(\ell,\ell') = -\frac{1}{2}\alpha i\Sigma(\ell,\ell') \tag{50}$$

This allows us, with $p = \alpha N$ and in anticipation of our expected time scaling $\delta_N = \tilde{\eta}/2p$ (known from the analysis in [10] of the Markovian limit $\zeta = 1$), to write the remaining equations (39,40) in the simple form

$$R(\ell,\ell') = \lim_{N\to\infty} \frac{\partial}{\partial A_e(\ell')} \left\{ \frac{\tilde{\eta}}{2p{\delta_N}^2} \left\langle \langle \overline{W}[\ell',\ell;A,Z]A(\ell)\rangle_{\{A|Z\}} \right\rangle_{\{Z\}} \right\} \tag{51}$$

$$\Sigma(\ell,\ell') = \lim_{N\to\infty} \left\{ \frac{\tilde{\eta}^2}{2p{\delta_N}^2} \left\langle \langle \overline{W}[\ell,\ell';A,Z]A(\ell)A(\ell')\rangle_{\{A|Z\}} \right\rangle_{\{Z\}} \right\} \tag{52}$$

We see that R defines a response function associated with external bid perturbation, and hence obeys causality: $R(\ell,\ell') = 0$ for $\ell' > \ell$. This, in turn, enables us to simplify equations (42) for $\{C,G\}$ and the measure $\langle\ldots\rangle_\star$ to a form identical to that found in [10] for the Markovian ('fake history') on-line MG:

$$C(t,t') = \langle \sigma[q(t),z(t)]\sigma[q(t'),z(t')]\rangle_\star \tag{53}$$

$$G(t,t') = \frac{\delta}{\delta\tilde{\theta}(t')}\langle \sigma[q(t),z(t)]\rangle_\star \tag{54}$$

$$\langle g[\{q,z\}]\rangle_\star = \frac{\int\{dq\}\ \langle g[\{q,z\}]\ M[\{q,z\}]\rangle_{\mathbf{z}}}{\int\{dq\}\ \langle M[\{q,z\}]\rangle_{\mathbf{z}}} \tag{55}$$

$$M[\{q,z\}] = p_0(q(0)) \int\{d\hat{q}\}\ e^{-\frac{1}{2}\alpha\int dt dt'\ \Sigma(t,t')\hat{q}(t)\hat{q}(t')} \times\ e^{i\int dt\ \hat{q}(t)\left[\frac{d}{dt}q(t)-\tilde{\theta}(t)+\alpha\int dt'\ R(t,t')\sigma[q(t'),z(t')]\right]} \tag{56}$$

4.2. Summary and interpretation

We recognize that (56) describes the usual effective single-trader equation with a retarded self-interaction and zero-average Gaussian noise $\eta(t)$ with covariances $\langle\eta(t)\eta(t')\rangle = \Sigma(t,t')$:

$$\frac{d}{dt}q(t) = \tilde{\theta}(t) - \alpha\int_0^t dt'\ R(t,t')\ \sigma[q(t')] + \sqrt{\alpha}\ \eta(t) \tag{57}$$

We have used the fact, as in [10], that the discontinuity of the correlation function for equal times, i.e. $C(t,t) = 1$, will in the continuous time limit be irrelevant. This implies that we may carry out the averages over the decision noise and are left only with expressions involving $\sigma[q] = \int dz\ P(z)\sigma[q,z]$, and that (with the exclusion of $t = t'$, where one has $C(t,t) = 1$) the order parameter equations (53,54) simplify to

$$C(t,t') = \langle\sigma[q(t)]\sigma[q(t')]\rangle_\star \qquad G(t,t') = \frac{\delta}{\delta\tilde{\theta}(t')}\langle\sigma[q(t)]\rangle_\star \tag{58}$$

Our remaining problem is to solve the order parameters $\{R,\Sigma\}$ from (51,52). To do so we must select the canonical time scale δ_N such that the $N\to\infty$ limit in

(51,52) is both non-trivial (i.e. δ_N sufficiently small) and well-defined (i.e. δ_N not too small). For the special value $\zeta = 1$ we know [10] that $\delta_N = \tilde{\eta}/2p$. Although here we have followed a different route towards a continuous time description, we show in Appendix A that indeed $\delta_N = \tilde{\eta}/2p$, by working out our present equations in detail for the fake history limit $\zeta \to 1$. Given this canonical time scaling and given the definition $\overline{W}[\ell, \ell'; A, Z] = \delta_{\boldsymbol{\lambda}(\ell,A,Z),\boldsymbol{\lambda}(\ell',A,Z)}$, we find our equations (51,52) taking their final forms:

$$R(t,t') = \lim_{\delta_N \to 0} \frac{\delta}{\delta A_e(t')} \left\langle\!\!\left\langle A(\ell)\delta_{\boldsymbol{\lambda}(\ell,A,Z),\boldsymbol{\lambda}(\ell',A,Z)} \right\rangle\!\!\right\rangle_{\{A,Z\}}\bigg|_{\ell=t/\delta_N,\ell'=t'/\delta_N} \tag{59}$$

$$\Sigma(t,t') = \tilde{\eta} \lim_{\delta_N \to 0} \frac{1}{\delta_N} \left\langle\!\!\left\langle A(\ell)A(\ell')\delta_{\boldsymbol{\lambda}(\ell,A,Z),\boldsymbol{\lambda}(\ell',A,Z)} \right\rangle\!\!\right\rangle_{\{A,Z\}}\bigg|_{\ell=t/\delta_N,\ell'=t'/\delta_N} \tag{60}$$

with $\delta/\delta A_e(\ell) = {\delta_N}^{-1}\partial/\partial A(\ell)$. Here $\langle\langle \ldots \rangle\rangle_{A,Z}$ refers to an average over the stochastic process (49) for the overall bids $\{A\}$ and over the pseudo-history $\{Z\}$. The bid evolution process can be written in more explicit form as

$$A(\ell) = A_e(\ell) + \phi_\ell - \frac{1}{2}\tilde{\eta}\sum_{\ell'<\ell} G(\ell,\ell')\delta_{\boldsymbol{\lambda}(\ell,A,Z),\boldsymbol{\lambda}(\ell',A,Z)}\ A(\ell') \tag{61}$$

with the zero-average Gaussian random fields $\{\phi\}$, characterized by

$$\langle \phi_\ell \phi_{\ell'} \rangle_{\{\phi|A,Z\}} = \frac{1}{2}[1 + C(\ell,\ell')]\ \delta_{\boldsymbol{\lambda}(\ell,A,Z),\boldsymbol{\lambda}(\ell',A,Z)} \tag{62}$$

Equation (61) is to be interpreted as follows. For every realization $\{Z\}$ of the fake history 'path' one iterates (61) to find successive bid values upon generating the zero-average Gaussian random variables ϕ_ℓ with statistics (62) (which depend, in turn, on the recent bid realizations). The result is averaged over the fake history paths $\{Z\}$.

Let us now summarize the structure of the present theory describing the MG with true market history in the limit $N \to \infty$, by indicating the similarities and the differences with the previous theory describing the on-line MG without market history:

similarities between the theory of real and fake history MGs:

- The MG with real history is described again by the effective single agent equation (57), from which the usual order dynamical order parameters $\{C, G\}$ are to be solved self-consistently via (58).
- The scaling with N of the characteristic times in the MG with history is identical to that of the MG without history, if we avoid highly biased global bid initializations (where the MG with history acts faster by a factor $\sqrt{N}$).

differences between the theory of real and fake history MGs:

- Real and fake history MGs differ in the retarded self-interaction kernel R and the noise covariance kernel Σ of the single agent equation. Without history, $\{R, \Sigma\}$ were found as explicit functions of $\{C, G\}$. With history they are to be solved from an effective equation (61) for the evolving global bid.

the effective global bid process:

- The effective global bid process (61) is itself independent of the stochastic effective single trader process (57). The two are linked only via the (time dependent) order parameters occurring in their definitions.
- At each stage in the process (61), the bid $A(\ell)$ is coupled directly only to bids in the past at times ℓ' with *identical realization of the M-bit history string.* In addition, only those effective global bid noise variables ϕ_ℓ are correlated which correspond to times ℓ with identical realizations of the M-bit history string.

The differences between the two 'fake history' definitions (3,4) (i.e. consistent versus inconsistent) are seen to be limited to the details of the averaging process $\langle \ldots \rangle_{\{Z\}}$.

In Appendix A we show how one can recover from (57,61) the earlier theory of [10] in the fake history limit $\zeta \to 1$. This exercise serves two purposes. Firstly, it confirms that the canonical time scale of our process is indeed given by $\delta_N = \tilde{\eta}/2p$ (modulo an irrelevant multiplicative constant). More importantly, being the simplest instance of our presently studied class of MG models, it provides useful intuition on how we might proceed to find solutions of our effective processes (57,61) in the general case.

5. The role of history statistics

We continue with our analysis of the full MG with history, and next show that all the effects induced by having real market history can be concentrated in the statistics of the M-bit memory strings $\boldsymbol{\lambda}$ of (15). More specifically, the core objects in the theory will turn out to be the following functions, which measure the joint probability to find identical histories in the effective global bid process (61) at k specified times $\{\ell_1, \ldots, \ell_k\}$, relative to the probability p^{-k} for this to happen in the case of randomly drawn fake histories and non-identical times:

$$\Delta_k(\ell_1, \ldots, \ell_k) = p^{k-1} \sum_{\boldsymbol{\lambda}} \langle\!\langle \prod_{i=1}^{k} \delta_{\boldsymbol{\lambda}, \boldsymbol{\lambda}(\ell_i, A, Z)} \rangle\!\rangle_{\{A,Z\}} \tag{63}$$

We have abbreviated $\sum_{\boldsymbol{\lambda}} = \sum_{\boldsymbol{\lambda} \in \{-1,1\}^M}$, with $2^M = p = \alpha N$. For any value of k one recovers in the random history limit and for non-identical times $\lim_{\zeta \to 1} \Delta_k(\ldots) = 1$. For $k = 1$ one has $\Delta_1(\ell) = \sum_{\boldsymbol{\lambda}} \langle\!\langle \delta_{\boldsymbol{\lambda}, \boldsymbol{\lambda}(\ell, A, Z)} \rangle\!\rangle_{\{A,Z\}} = 1$, for any ζ. In contrast, for arbitrary ζ (i.e. when allowing for real histories) and $k > 1$ the functions (63) are nontrivial.

5.1. Reduction of the kernels $\{R, \Sigma\}$

We will follow as much as possible the steps which we took in Appendix A in order to recover the $\zeta = 1$ equations (A.9,A.14). We re-write the global bid equation (61) as

$$\sum_{\ell' \leq \ell} \Big\{ \delta_{\ell\ell'} + \frac{1}{2} \tilde{\eta} G(\ell, \ell') \delta_{\boldsymbol{\lambda}(\ell, A, Z), \boldsymbol{\lambda}(\ell', A, Z)} \Big\} A(\ell') = A_e(\ell) + \phi_\ell$$

and we formally invert the operator on the left-hand side, using $\delta_N = \tilde{\eta}/2p$:

$$A(\ell) = A_e(\ell) + \phi_\ell + \sum_{r>0} (-\frac{\tilde{\eta}}{2})^r \sum_{\ell_1 \ldots \ell_r} G(\ell, \ell_1) G(\ell_1, \ell_2) \ldots G(\ell_{r-1}, \ell_r) \\ \times \Big[\prod_{i=1}^{r} \delta_{\boldsymbol{\lambda}(\ell,A,Z), \boldsymbol{\lambda}(\ell_i,A,Z)} \Big] \left[A_e(\ell_r) + \phi_{\ell_r} \right] \tag{64}$$

Expression (64) is itself not yet a solution of (61), since the bids $\{A(s)\}$ also occur inside the history strings $\boldsymbol{\lambda}(\ell', A, Z)$ at the right-hand side. We now insert (64) first into (59), and consider only infinitesimal external bid perturbations A_e, so that we need not worry about indirect effects on $A(\ell)$ of these perturbations via the history strings $\boldsymbol{\lambda}(s, A, Z)$:

$$R(t, t') = \delta(t - t') + \lim_{\delta_N \to 0} \Big\{ \sum_{r>0} (-\delta_N)^{r-1} \sum_{\ell_1 \ldots \ell_{r-1}} G(\ell, \ell_1) G(\ell_1, \ell_2) \ldots G(\ell_{r-1}, \ell') \\ \times p^r \Big\langle\!\Big\langle \delta_{\boldsymbol{\lambda}(\ell,A,Z), \boldsymbol{\lambda}(\ell',A,Z)} \prod_{i=1}^{r-1} \delta_{\boldsymbol{\lambda}(\ell,A,Z), \boldsymbol{\lambda}(\ell_i,A,Z)} \Big\rangle\!\Big\rangle_{\{A,Z\}} \Big\} \Big|_{\ell = \frac{t}{\delta_N}, \ell' = \frac{t'}{\delta_N}} \\ = \delta(t - t') + \lim_{\delta_N \to 0} \Big\{ \sum_{r>0} (-\delta_N)^{r-1} \sum_{\ell_1 \ldots \ell_{r-1}} G(\ell_0, \ell_1) \ldots G(\ell_{r-1}, \ell_r) \\ \times \Delta_{r+1}(\ell_0, \ldots, \ell_r) \Big\} \Big|_{\ell_0 = \frac{t}{\delta_N}, \ell_r = \frac{t'}{\delta_N}} \tag{65}$$

Similarly we can insert (64) into (60), again with $A_e \to 0$, and find

$$\Sigma(t, t') = \tilde{\eta} \lim_{\delta_N \to 0} \frac{1}{\delta_N} \Big\{ \sum_{r,r' \geq 0} (-\delta_N)^{r+r'} \sum_{\ell_1 \ldots \ell_r} G(\ell_0, \ell_1) \ldots G(\ell_{r-1}, \ell_r) \\ \times \sum_{\ell'_1 \ldots \ell'_r} G(\ell'_0, \ell'_1) \ldots G(\ell'_{r-1}, \ell'_r) \; p^{r+r'} \Big\langle\!\Big\langle \; \langle \phi_{\ell_r} \phi_{\ell'_{r'}} \rangle_{\{\phi|A,Z\}} \\ \times \Big[\prod_{i=1}^{r} \delta_{\boldsymbol{\lambda}(\ell_0,A,Z), \boldsymbol{\lambda}(\ell_i,A,Z)} \Big] \Big[\prod_{j=1}^{r'} \delta_{\boldsymbol{\lambda}(\ell'_0,A,Z), \boldsymbol{\lambda}(\ell'_j,A,Z)} \Big] \; \Big\rangle\!\Big\rangle_{\{A,Z\}} \Big\} \Big|_{\ell_0 = \frac{t}{\delta_N}, \ell'_0 = \frac{t'}{\delta_N}} \\ = \lim_{\delta_N \to 0} \Big\{ \sum_{r,r' \geq 0} (-\delta_N)^{r+r'} \sum_{\ell_1 \ldots \ell_r} G(\ell_0, \ell_1) \ldots G(\ell_{r-1}, \ell_r) \\ \times \sum_{\ell'_1 \ldots \ell'_r} G(\ell'_0, \ell'_1) \ldots G(\ell'_{r-1}, \ell'_r) \; [1 + C(\ell_r, \ell'_{r'})] \\ \times \Delta_{r+r'+2}(\ell_0, \ldots, \ell_r, \ell'_0, \ldots, \ell'_{r'}) \Big\} \Big|_{\ell_0 = \frac{t}{\delta_N}, \ell'_0 = \frac{t'}{\delta_N}} \tag{66}$$

The limits $\delta_N \to 0$ in (65,66) are well-defined, since each time summation combines with a factor δ_N to generate an integral, whereas pairwise identical times in (66) leave a 'bare' factor δ_N but will also cause $\Delta_{r+r'+2}(\ldots)$ to gain a factor $p = \tilde{\eta}/2\delta_N$ in compensation.

Since the single agent process (57) is linked to the global bid process (61) only via the kernels $\{R, \Sigma\}$, we conclude from (65,66) (which are still fully exact) that the effects of having true market history are concentrated solely in the resulting history statistics as described by the functions (63). More specifically, there is no need for us to solve the global bid process (61) beyond knowing the history statistics which it generates.

5.2. Time-translation invariant stationary states

In fully ergodic and time-translation invariant states without anomalous response, we could in 'fake history' MG versions find exact closed equations for persistent order parameters without having to solve for the kernels $\{C, G\}$ in full, and locate phase transitions exactly. This suggests that the same may be true for MGs with true history. Thus we make the standard time-translation invariance (TTI) ansatz for the kernels in (57) and for the correlation- and response functions:

$$C(t,t') = C(t-t') \qquad G(t,t') = G(t-t')$$
$$R(t,t') = R(t-t') \qquad \Sigma(t,t') = \Sigma(t-t')$$

with $\chi = \int_0^\infty dt\ R(t)$ finite. It turns out that several relations between persistent observables in TTI stationary states of the present non-Markovian MG process, if such states again exist, can be established on the basis of (57) alone. Upon following established notation conventions and abbreviating time averages as $\overline{f} = \lim_{\tau\to\infty} \tau^{-1} \int_0^\tau dt\ f(t)$, we may write the time average of (57) as

$$\overline{dq/dt} = \overline{\tilde{\theta}} - \alpha\chi_R\ \overline{\sigma} + \sqrt{\alpha}\ \overline{\eta} \tag{67}$$

with $\chi_R = \int_0^\infty dt\ R(t)$. We may now define the familiar effective agent trajectories corresponding to fickle versus frozen agents as those with either $\overline{dq/dt} = 0$ or $\overline{dq/dt} \neq 0$, respectively. For frozen agents, consistency demands that $\text{sgn}[\overline{\sigma}] = \text{sgn}[\overline{dq/dt}]$. It then follows from (67) that the (at least for $\chi_R > 0$ complementary and mutually exclusive) conditions for having a 'fickle' or a 'frozen' solution can be written as follows:

$$\text{fickle}: \quad |\overline{\theta} + \sqrt{\alpha}\overline{\eta}| \leq \alpha\chi_R\sigma[\infty], \quad \overline{\sigma} = \frac{\overline{\tilde{\theta}} + \sqrt{\alpha}\ \overline{\eta}}{\alpha\chi_R} \tag{68}$$

$$\text{frozen}: \quad |\overline{\theta} + \sqrt{\alpha}\overline{\eta}| > \alpha\chi_R\sigma[\infty], \quad \overline{\sigma} = \sigma[\infty].\text{sgn}\Big[\frac{\overline{\tilde{\theta}} + \sqrt{\alpha}\ \overline{\eta}}{\alpha\chi_R}\Big] \tag{69}$$

Which solution of (68) and (69) we will find depends on the realization of the noise term $\overline{\eta}$, which is a frozen Gaussian variable with zero expectation value and with variance

$$S_0^2 = \langle \overline{\eta}^2 \rangle_\star = \lim_{\tau\to\infty} \frac{1}{\tau^2} \int_0^\tau dt dt'\ \Sigma(t,t') = \Sigma(\infty) \tag{70}$$

We may now proceed as in Appendix A towards the calculation of the persistent order parameters ϕ, χ and c, where ϕ denotes the fraction of frozen agents in the stationary state, where $\chi = \int_0^\infty dt\ G(t)$, and with

$$c = \lim_{t\to\infty} C(t) = \lim_{\tau\to\infty} \frac{1}{\tau^2} \int_0^\tau dt dt'\ \langle \sigma[q(t)]\sigma[q(t')] \rangle_\star = \langle \overline{\sigma}^2 \rangle_\star \tag{71}$$

Upon introducing the short-hand $u = \sqrt{\alpha}\chi_R\sigma[\infty]/S_0\sqrt{2}$, and upon using the conditions and relations (68,69), we find in the limit $\tilde{\theta} \to 0$ of vanishing external fields:

$$\begin{aligned} \phi &= \int \frac{d\overline{\eta}}{S_0\sqrt{2\pi}}\ e^{-\frac{1}{2}\overline{\eta}^2/S_0^2} \theta\Big[|\overline{\eta}| - \sqrt{\alpha}\chi_R\sigma[\infty]\Big] \\ &= 1 - \text{Erf}[u] \end{aligned} \tag{72}$$

$$
\begin{aligned}
c &= \int \frac{d\overline{\eta}}{S_0\sqrt{2\pi}}\; e^{-\frac{1}{2}\overline{\eta}^2/S_0^2}\Big\{\theta\Big[|\overline{\eta}|-\sqrt{\alpha}\chi_{R}\sigma[\infty]\Big]\sigma^2[\infty] \\
&\qquad\qquad + \theta\Big[\sqrt{\alpha}\chi_{R}\sigma[\infty]-|\overline{\eta}|\Big]\frac{\overline{\eta}^2}{\alpha\chi_{R}^2}\Big\} \\
&= \sigma^2[\infty]\Big\{1-\mathrm{Erf}[u]+\frac{1}{2u^2}\mathrm{Erf}[u]-\frac{1}{u\sqrt{\pi}}e^{-u^2}\Big\} \qquad (73)
\end{aligned}
$$

$$
\begin{aligned}
\chi &= \int \frac{d\overline{\eta}}{S_0\sqrt{2\pi}}\; e^{-\frac{1}{2}\overline{\eta}^2/S_0^2}\frac{\partial\overline{\sigma}}{\partial(\sqrt{\alpha}\overline{\eta})} \\
&= \mathrm{Erf}[u]/\alpha\chi_{R} \qquad (74)
\end{aligned}
$$

Hence, in order to find the TTI stationary solution $\{\phi, c, \chi\}$ and the phase transition point (defined by $\chi \to \infty$), we only need to extract expressions for χ_{R} and S_0 from the stochastic overall bid process (61). Using (65,66), the latter can be written as

$$
\begin{aligned}
\chi_{R} &= \int_0^\infty dt\; R(t) \\
&= 1+\lim_{\delta_N\to 0}\Big\{\sum_{r>0}(-\delta_N)^r\sum_{\ell_1\ldots\ell_r}G(\ell_1-\ell_2)G(\ell_2-\ell_3)\ldots G(\ell_{r-1}-\ell_r)G(\ell_r) \\
&\qquad\qquad \times\; \Delta_{r+1}(\ell_1,\ldots,\ell_r,0)\Big\} \qquad (75)
\end{aligned}
$$

$$
\begin{aligned}
S_0^2 &= \lim_{L\to\infty}\frac{1}{L^2}\sum_{\ell_0,\ell_0'\le L}\lim_{\delta_N\to 0}\Bigg\{\sum_{r,r'\ge 0}(-\delta_N)^{r+r'}\sum_{\ell_1\ldots\ell_r}G(\ell_0-\ell_1)\ldots G(\ell_{r-1}-\ell_r) \\
&\qquad \times\sum_{\ell_1'\ldots\ell_r'}G(\ell_0'-\ell_1')\ldots G(\ell_{r-1}'-\ell_r')\;[1+C(\ell_r-\ell_{r'}')] \\
&\qquad \times\;\Delta_{r+r'+2}(\ell_0,\ldots,\ell_r,\ell_0',\ldots,\ell_{r'}')\Bigg\} \qquad (76)
\end{aligned}
$$

5.3. TTI states with short history correlation times

Calculating the history statistics kernels (63) from the global bid process (61) is hard, but in those cases where the history correlation time L_h (measured in individual iterations ℓ) in the process is much smaller than N, we can make progress in our analysis of TTI stationary states. We define the asymptotic frequency $\pi_{\boldsymbol{\lambda}}(A,Z)$ at which history string $\boldsymbol{\lambda}$ occurs in a given realization $\{A,Z\}$ of our process (61) as

$$
\pi_{\boldsymbol{\lambda}}(A,Z) = \lim_{L\to\infty}\frac{1}{L}\sum_{\ell=1}^{L}\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ell,A,Z)} \qquad (77)
$$

Obviously $\sum_{\boldsymbol{\lambda}}\pi_{\boldsymbol{\lambda}}(A,Z)=1$. For $\zeta=1$ (no history) we would have $\pi_{\boldsymbol{\lambda}}=p^{-1}$ for all $\boldsymbol{\lambda}$. We may also define the distribution $\varrho(f)$ of these asymptotic history frequencies $\pi_{\boldsymbol{\lambda}}(A,Z)$, relative to the benchmark 'no-memory' values p^{-1}, and averaged over the global bid process (61) in the infinite system size (i.e. continuous time) limit:

$$
\varrho(f) = \lim_{p\to\infty}\frac{1}{p}\sum_{\boldsymbol{\lambda}}\langle\!\langle\delta[f-p\pi_{\boldsymbol{\lambda}}(A,Z)]\rangle\!\rangle_{\{A,Z\}} \qquad (78)
$$

Our definitions guarantee that $\int_0^\infty df\; f\varrho(f)=1$ for any ζ. For $\zeta=1$ we simply recover $\varrho(f)=\delta[f-1]$, i.e. all histories occur equally frequently. We have not yet shown that

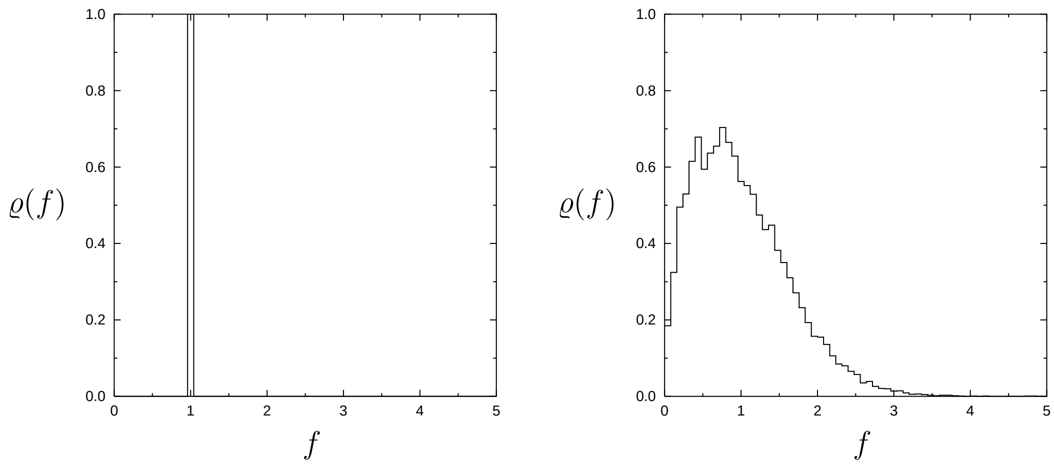

Figure 1. Typical examples of history frequency distributions (78) as measured in simulations of the on-line MG without decision noise but with full history (i.e. $\zeta = 0$), after equilibration. Here $N = 8193$. Left: $\alpha = 0.125$ (in the non-ergodic regime of the MG, below α_c). Right: $\alpha = 2.0$ (in the ergodic regime, above α_c).

the limit in (78) exists, i.e. that the history frequencies do indeed generally scale as $\pi_{\boldsymbol{\lambda}}(A, Z) = \mathcal{O}(N^{-1})$. Numerical simulations, however, confirm quite convincingly that this ansatz is indeed correct (see e.g. Figure 1).

If L_h is the history correlation time in the process (61), then *finite* samples of history occurrence frequencies can be expected to approach the asymptotic value (77) as

$$\frac{1}{2L} \sum_{\ell'=\ell-L}^{\ell+L} \delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ell',A,Z)} = \pi_{\boldsymbol{\lambda}}(A, Z) \left[1 + \mathcal{O}((L_h/L)^{\frac{1}{2}})\right] \tag{79}$$

This implies that in expressions such as (75), where $G(\ell) - G(\ell') = \mathcal{O}(|\ell - \ell'|/N)$ and where only time strings $\{\ell_1, \ldots, \ell_k\}$ with mutual temporal separations of order $\mathcal{O}(N)$ will survive the limit $\delta_N \to 0$, we may choose e.g. $L = \sqrt{L_h N}$ and effectively replace

$$\Delta_{r+1}(\ell_1, \ldots, \ell_r, 0) \;\to\; p^r \sum_{\boldsymbol{\lambda}} \left[\pi_{\boldsymbol{\lambda}}[1 + \mathcal{O}(\sqrt{L_h/N})]\right]^{r+1} \tag{80}$$

This results in

$$\begin{aligned} \chi_R &= \lim_{p\to\infty} \frac{1}{p} \sum_{\boldsymbol{\lambda}} \sum_{r\geq 0} (-\chi)^r \left[p\pi_{\boldsymbol{\lambda}}[1 + \mathcal{O}(\sqrt{L_h/L})]\right]^{r+1} \\ &= \lim_{p\to\infty} \frac{1}{p} \sum_{\boldsymbol{\lambda}} \frac{p\pi_{\boldsymbol{\lambda}}[1 + \mathcal{O}(\sqrt{L_h/N})]}{1 + \chi\; p\pi_{\boldsymbol{\lambda}}[1 + \mathcal{O}(\sqrt{L_h/N})]} = \int_0^\infty df\; \varrho(f)\; \frac{f}{1 + \chi f} \end{aligned} \tag{81}$$

(provided indeed $\lim_{N\to\infty} L_h/N = 0$). The same simplification to an expression involving only the distribution $\varrho(f)$ can be achieved in (76), but there we have to be more careful in dealing with the occurrences of similar or identical times in the argument of (63). We first rewrite (76) by transforming the iteration times according to

$$\text{for all} \;\; i \in \{0, \ldots, r\}: \qquad \ell_i = \sum_{j=i}^{r} s_j$$

This gives, using $\lim_{s\to\infty} G(s) = 0$ (i.e. restricting ourselves to ergodic states with normal response):

$$S_0^2 = \lim_{\delta_N\to 0} \sum_{r,r'\geq 0} (-\delta_N)^{r+r'} \sum_{s_0\ldots s_{r-1}>0} G(s_0)\ldots G(s_{r-1}) \sum_{s'_0\ldots s'_{r'-1}>0} G(s'_0)\ldots G(s'_{r-1})$$
$$\times \lim_{L\to\infty} \frac{1}{L^2} \sum_{s_r=0}^{L-\sum_{i=0}^{r-1} s_i} \sum_{s'_{r'}=0}^{L-\sum_{i=0}^{r'-1} s'_i} [1+C(s_r - s'_{r'})]$$
$$\times \ \Delta_{r+r'+2}(s_0+\ldots+s_r, \ldots, \ell_{r-1}+\ell_r, \ell_r, \ell'_0+\ldots+\ell_{r'}, \ldots, \ell'_{r'-1}+\ell'_{r'}, \ell'_{r'})$$

Each time summation is compensated either by a factor δ_N (giving an integral), or limited in range by L and compensated by an associated factor L^{-1}, so that any 'pairing' where two (or more) times are close to each other (relative to the correlation time L_h) will not survive the combined limits $\delta_N \to 0$ and $L \to \infty$. Thus we may again put

$$\Delta_{r+r'+2}(\ldots\ldots) \ \to \ p^{r+r'+1} \sum_{\boldsymbol{\lambda}} \Big[\pi_{\boldsymbol{\lambda}}[1+\mathcal{O}(\sqrt{L_h/N})]\Big]^{r+r'+2} \tag{82}$$

and find, with $C(\infty) = c$:

$$\begin{aligned}
S_0^2 &= (1+c) \lim_{p\to\infty} \frac{1}{p} \sum_{\boldsymbol{\lambda}} \sum_{r,r'\geq 0} (-\chi)^{r+r'} \Big[p\pi_{\boldsymbol{\lambda}}[1+\mathcal{O}(\sqrt{L_h/N})]\Big]^{r+r'+2} \\
&= (1+c) \lim_{p\to\infty} \frac{1}{p} \sum_{\boldsymbol{\lambda}} \frac{\Big[p\pi_{\boldsymbol{\lambda}}[1+\mathcal{O}(\sqrt{L_h/N})]\Big]^2}{\Big[1+\chi\, p\pi_{\boldsymbol{\lambda}}[1+\mathcal{O}(\sqrt{L_h/N})]\Big]^2} \\
&= (1+c) \int_0^\infty df \ \varrho(f) \ \frac{f^2}{(1+\chi f)^2}
\end{aligned} \tag{83}$$

Since only χ_R and S_0 are needed to solve our effective single agent process in TTI stationary states, we see that upon making the ansatz of short history correlation times $L_h \ll N$ the effects of history on the persistent order parameters in the MG are fully concentrated in the distribution $\varrho(f)$ of history frequencies, as defined by (78). Once $\varrho(f)$ has been extracted from the process (61), the TTI order parameters are given by the solution of the following set of equations:

$$u = \frac{\sigma[\infty]\sqrt{\alpha}\chi_R}{S_0\sqrt{2}} \qquad \chi = \frac{1-\phi}{\alpha\chi_R} \qquad \phi = 1 - \mathrm{Erf}[u] \tag{84}$$

$$c = \sigma^2[\infty]\Big\{1 - \mathrm{Erf}[u] + \frac{1}{2u^2}\mathrm{Erf}[u] - \frac{1}{u\sqrt{\pi}} e^{-u^2}\Big\} \tag{85}$$

$$\chi_R = \int_0^\infty df \ \varrho(f) \ \frac{f}{1+\chi f} \tag{86}$$

$$S_0^2 = (1+c) \int_0^\infty df \ \varrho(f) \ \frac{f^2}{(1+\chi f)^2} \tag{87}$$

For $\zeta = 1$ (the fake history limit) we have $\varrho(f) = \delta[f-1]$, leading to $\chi_R = (1+\chi)^{-1}$ and $S_0 = \sqrt{1+c}/(1+\chi)$, and the above equations are seen to reduce to the corresponding ones in [10], as they should.

6. Calculating the history statistics

Upon making the ansatz of short history correlation times in the MG, we have shown that finding closed equations for persistent TTI order parameters boils down to calculating the distribution $\varrho(f)$ of relative history frequencies, as defined in (78)‖. Our remaining programme of analysis is: (i) finding an expression for $\varrho(f)$, (ii) expressing this distribution in terms of the persistent order parameters $\{c, \phi, \chi, \chi_R, S_0\}$, and (iii) confirming retrospectively the consistency of assuming short history correlation times.

6.1. The moments of $\varrho(f)$

The distribution (78) is generated by the non-Markovian process (61), which we cannot hope to solve directly. However, we can get away with a self-consistent calculation which does not require solving (61) in full. We focus on the moments μ_k of the distribution ϱ, from which the latter can always be recovered (if the integrals below exist):

$$\mu_k \;=\int_0^\infty df\;\varrho(f)f^k \tag{88}$$

$$\varrho(f)=\int\frac{d\omega}{2\pi}e^{i\omega f}\sum_{k\geq 0}\frac{\mu_k}{k!}(-i\omega)^k \tag{89}$$

Obviously $\mu_0=\mu_1=1$, for any ζ, which follows directly from definition (78). In the absence of history (i.e. $\zeta=1$) we have $\varrho(f)=\delta[f-1]$, so that $\mu_k=1$ for all $k\geq 0$. We will rely on the sum over moments in (89) converging on scales of k which are independent of N. This is equivalent to saying that the limit (78) is well-defined, so it does not restrict us further. By combining the definitions (77,78,88) and (15), we can obtain a more explicit but still relatively simple expression for the moments μ_k:

$$\begin{aligned}\mu_k&=\frac{1}{p}\sum_{\boldsymbol{\lambda}}\langle\!\langle[p\pi_{\boldsymbol{\lambda}}(A,Z)]^k\rangle\!\rangle_{\{A,Z\}}\\&=\lim_{L\to\infty}\frac{p^k}{L^k}\sum_{\ell_1\ldots\ell_k=1}^{L}\Big\langle\!\Big\langle\prod_{i=1}^{M}\Big\{\frac{1}{2}\sum_{\lambda=\pm1}\prod_{j=1}^{k}\delta_{\lambda,\lambda_i(\ell_j,A,Z)}\Big\}\Big\rangle\!\Big\rangle_{\{A,Z\}}\\&=\lim_{L\to\infty}\frac{p^{k-1}}{L^k}\sum_{\ell_1\ldots\ell_k=1}^{L}\Big\langle\!\Big\langle\prod_{i=1}^{M}\Big\{\prod_{j=1}^{k}\delta_{1,\lambda_i(\ell_j,A,Z)}+\prod_{j=1}^{k}\delta_{-1,\lambda_i(\ell_j,A,Z)}\Big\}\Big\rangle\!\Big\rangle_{\{A,Z\}}\end{aligned} \tag{90}$$

The average $\langle\langle\ldots\rangle\rangle_{\{A,Z\}}$ in the last line of (90) equals the following joint probability:

$$\begin{aligned}\langle\langle\ldots\rangle\rangle_{\{A,Z\}}=\text{Prob}\Big[\;&\Big\{\lambda_1(\ell_1,A,Z)=\lambda_1(\ell_2,A,Z)=\;\ldots\;=\lambda_1(\ell_k,A,Z)\Big\}\\\text{and}\;&\Big\{\lambda_2(\ell_1,A,Z)=\lambda_2(\ell_2,A,Z)=\;\ldots\;=\lambda_2(\ell_k,A,Z)\Big\}\\&\qquad\vdots\\\text{and}\;&\Big\{\lambda_M(\ell_1,A,Z)=\lambda_M(\ell_2,A,Z)=\;\ldots\;=\lambda_M(\ell_k,A,Z)\Big\}\Big]\end{aligned} \tag{91}$$

‖ A similar conclusion was reached also in [15], but on the basis of several approximations. Furthermore, in contrast to the present GFA approach, in [15] there was no way to calculate $\varrho(f)$ from the theory.

Let us define the short-hand

$$Same(i) = \Big\{\lambda_i(\ell_1, A, Z) = \lambda_i(\ell_2, A, Z) = \ldots = \lambda_i(\ell_k, A, Z)\Big\} \tag{92}$$

which states that the i-th component of the history string takes the same value at the k specified times $\{\ell_1, \ldots, \ell_k\}$. Given that our bid process obeys causality¶, statement (91) can be written as

$$\begin{aligned}
\langle\langle\ldots\rangle\rangle_{\{A,Z\}} &= \mathrm{Prob}\Big[\ Same(1)\ \wedge\ Same(2)\ \wedge\ \ldots\ \wedge\ Same(M)\ \Big] \\
&= \quad \mathrm{Prob}[Same(1)\mid Same(2)\wedge\ldots\wedge Same(M)] \\
&\quad\ \times\ \mathrm{Prob}[Same(2)\mid Same(3)\wedge\ldots\wedge Same(M)] \\
&\qquad\qquad \vdots \\
&\quad\ \times\ \mathrm{Prob}[Same(M-1)\mid Same(M)] \\
&\quad\ \times\ \mathrm{Prob}[Same(M)]
\end{aligned} \tag{93}$$

Since we need not consider values of k which scale with N or L, the contributions to (90) from those times $\{\ell_1, \ldots, \ell_k\}$ for which there are correlations between objects at a time ℓ_r and those at another time $\ell_{r'}$ will vanish in the limit $L \to \infty$. Since we also know that we are in a TTI state, it follows that the conditional probabilities in (93) will not depend on the actual values $\{\ell_1, \ldots, \ell_k\}$. In the limit $L \to \infty$ we may replace

$$\mathrm{Prob}[Same(r)\mid Same(r+1)\wedge\ldots\wedge Same(M)]\ \to\ \mathcal{P}_{[k|M-r]}$$

where $\mathcal{P}_{[k|m]}$ denotes the probability to find for randomly drawn and infinitely separated times $\{\ell_1, \ldots, \ell_k\}$ that $\lambda_i(\ell_1, A, Z) = \ldots = \lambda_i(\ell_k, A, Z)$, for an index i, given that the identity holds for the indices $\{i+1, \ldots, i+m\}$ (with $\mathcal{P}_{[k|0]}$ giving this probability in the absence of conditions). This allows us to write (90) as

$$\mu_k = p^{k-1}\ \mathcal{P}_{[k|M-1]}.\mathcal{P}_{[k|M-2]}\ \ldots\ \mathcal{P}_{[k|1]}.\mathcal{P}_{[k|0]} \tag{94}$$

As a simple test one may verify (94) for the trivial case $\zeta = 1$ (fake history only). Here conditioning on the past is irrelevant, so $\mathcal{P}_{[k|m]} = \mathcal{P}_{[k|0]} = 2^{1-k}$ for all m, which indeed gives us $\mu_k = p^{k-1}2^{(1-k)M} = 1$ (as it should). In the continuous time limit $N \to \infty$ (equivalently: for $M \to \infty$, since $2^M = \alpha N$) we thus find the as yet exact formula

$$\lim_{M\to\infty} \log(\mu_k) = \lim_{M\to\infty} \sum_{r=0}^{M-1} \log\Big[2^{k-1}\mathcal{P}_{[k|r]}\Big] \tag{95}$$

6.2. Reduction to history coincidence statistics

Next we have to find an expression for the probabilities $\mathcal{P}_{[k|r]}$. We know from (61,62) that the value of the overall bid at any time ℓ is only correlated with the bid value at time ℓ' if the two times (ℓ, ℓ') have *identical* history strings, i.e. if $\boldsymbol{\lambda}(\ell, A, Z) = \boldsymbol{\lambda}(\ell', A, Z)$.

¶ We here use the fact that a component $\lambda_i(\ell, A, Z)$ of the history string observed by the agents at time ℓ is by construction (see definition (15)) referring to the overall bid at time $\ell - i$. It follows that the probability of finding a given value for $\lambda_i(\ell, A, Z)$ depends via causality only on the bids at the earlier times $\{\ell - i - 1, \ell - i - 2, \ldots\}$, hence on $\{\lambda_{i+1}(\ell, A, Z), \lambda_{i+2}(\ell, A, Z), \ldots\}$.

We know that individual histories show up during the process with probabilities of order N^{-1}. Since the likelihood of finding *recurring* histories during any number $r = \mathcal{O}(M)$ of consecutive iterations of our process is thus vanishingly small (of order $\mathcal{O}(M/N)$) such direct correlations are of no consequence in our calculation. The only relevant effect of conditioning in the sense of the $\mathcal{P}_{[k|r]}$ is via its biasing of histories in subsequent iterations. Although the probability of history recurrence during a time window of size $\mathcal{O}(M)$ is vanishingly small, if two (short) instances of global bid trajectories are found to have identical realizations of some of the bits of their history strings, they will nevertheless be more likely than average to have an *identical* history realization in the next time step. This is the subtle statistical effect which, together with the resulting biases in the bids which are subsequently found at times with specific histories, gives rise to the relative history frequency distributions $\varrho(f)$ as observed in e.g. Fig. 1.

The statement that the conditioning in $\mathcal{P}_{[k|r]}$ acts only via the joint likelihood of finding specific histories $\{\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_k\}$ at the k specified (and widely separated) times $\{\ell_1, \ldots, \ell_k\}$, translates into

$$\mathcal{P}_{[k|r]} = \sum_{\boldsymbol{\lambda}_1,\ldots\boldsymbol{\lambda}_k} \mathcal{P}[k|\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_k]\ \mathcal{P}[\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_k|r] \tag{96}$$

Here $\mathcal{P}[k|\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_k]$ denotes the likelihood to find $\boldsymbol{\lambda}(\ell_1, A, Z) = \ldots = \boldsymbol{\lambda}(\ell_k, A, Z)$, if the history strings at those k times equal $\{\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_k\}$, and $\mathcal{P}[\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_k|r]$ denotes the likelihood of finding those k specific histories given that the bits of the k history strings have been identical over the r most recent iterations$^+$. The probability of finding specific bid values $A(\ell)$ will in TTI states only depend on the history string $\boldsymbol{\lambda}$ associated with time ℓ. Given this history string, $A(\ell)$ is a Gaussian variable (this follows from the effective bid process (61)), with some average $\overline{A}_{\boldsymbol{\lambda}}$ and a variance $\sigma^2_{\boldsymbol{\lambda}}$ (which will in due course have to be calculated). Using also the fact that the $Z(\ell, i)$ were defined as zero average Gaussian variables, with variance κ^2, we obtain:

$$\begin{aligned}
\mathcal{P}[k|\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_k] = & \prod_{j=1}^{k} \Big[\int DZ \int dA\ P_{\boldsymbol{\lambda}_j}(A)\theta[(1-\zeta)A + \zeta Z]\Big] \\
& + \prod_{j=1}^{k} \Big[\int DZ \int dA\ P_{\boldsymbol{\lambda}_j}(A)\theta[-(1-\zeta)A - \zeta Z]\Big] \\
= & \prod_{j=1}^{k} \left[\frac{1}{2} + \frac{1}{2}\mathrm{Erf}\Big[\frac{(1-\zeta)\overline{A}_{\boldsymbol{\lambda}_j}}{\sqrt{2}\sqrt{\zeta^2\kappa^2 + (1-\zeta)^2\sigma^2_{\boldsymbol{\lambda}_j}}}\Big]\right] \\
& + \prod_{j=1}^{k} \left[\frac{1}{2} - \frac{1}{2}\mathrm{Erf}\Big[\frac{(1-\zeta)\overline{A}_{\boldsymbol{\lambda}_j}}{\sqrt{2}\sqrt{\zeta^2\kappa^2 + (1-\zeta)^2\sigma^2_{\boldsymbol{\lambda}_j}}}\Big]\right]
\end{aligned} \tag{97}$$

We now write the sum over all combinations of histories in (96) in terms of a partitioning

$^+$ Here one will find that consistent and inconsistent realizations of the history noise variables $Z(\ell, i)$ are to be treated differently: in the case of consistent noise, one will always have $\lambda_i(\ell, A, Z) = \lambda_{i+1}(\ell + 1, A, Z)$. This is not true for inconsistent history noise.

in groups, where two M-bit strings $\{\boldsymbol{\lambda}_i, \boldsymbol{\lambda}_j\}$ are in the same group if and only if they are identical. We write $(g_1, g_2, \ldots)$ for the subset of all combinations $\{\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_k\}$ with one group of size g_1, a second group of size g_2, and so on*. Clearly $g_1 + g_2 + \ldots = k$, for all possible subsets of our partitioning. This allows us to write

$$\mathcal{P}_{[k|r]} = \sum_{(g_1,g_2,\ldots)} \delta_{k,g_1+g_2+\ldots} \mathcal{P}[k|g_1, g_2, \ldots]\ \mathcal{P}[g_1, g_2, \ldots |r] \tag{98}$$

According to (97), the distribution $\mathcal{P}[k|g_1, g_2, \ldots]$ is of the relatively simple form $\mathcal{P}[k|g_1, g_2, \ldots] = 2^{1-k}\Phi(g_1, g_2, \ldots)$, with

$$\begin{aligned}\Phi(g_1, g_2, \ldots) = \frac{1}{2}\prod_{j\geq 1}\left\{\sum_{\boldsymbol{\lambda}} \pi_{\boldsymbol{\lambda}} \left[1 + \mathrm{Erf}\Big[\frac{(1-\zeta)\overline{A}_{\boldsymbol{\lambda}}}{\sqrt{2}\sqrt{\zeta^2\kappa^2 + (1-\zeta)^2\sigma^2_{\boldsymbol{\lambda}}}}\Big]\right]^{g_j}\right\} \\ + \frac{1}{2}\prod_{j\geq 1}\left\{\sum_{\boldsymbol{\lambda}} \pi_{\boldsymbol{\lambda}} \left[1 - \mathrm{Erf}\Big[\frac{(1-\zeta)\overline{A}_{\boldsymbol{\lambda}}}{\sqrt{2}\sqrt{\zeta^2\kappa^2 + (1-\zeta)^2\sigma^2_{\boldsymbol{\lambda}}}}\Big]\right]^{g_j}\right\}\end{aligned} \tag{99}$$

Insertion of the representation (98) for $\mathcal{P}_{[k|r]}$ into (95) allows us to write the moments of the relative history frequencies in the following form:

$$\lim_{M\to\infty} \log(\mu_k) = \lim_{M\to\infty} \sum_{r=0}^{M-1} \log\left[\sum_{(g_1,g_2,\ldots)} \delta_{k,g_1+g_2+\ldots}\Phi(g_1, g_2, \ldots)\mathcal{P}[g_1, g_2, \ldots |r]\right] \tag{100}$$

It will be helpful to assess which values of r in (100) can survive the limit $M \to \infty$. Whenever we have a value r such that $M-r \to \infty$ as $M \to \infty$, the condition that the k history bits were identical over the most recent r steps still leaves a large $\mathcal{O}(2^{M-r})$ number of compatible history strings to be found at the probing times $\{\ell_1, \ldots, \ell_k\}$, so the likelihood of finding histories coinciding in multiples $(g_1, g_2, \ldots)$ scales as

$$\mathcal{P}[g_1, g_2, \ldots |r] = \prod_{j|g_j>1} \mathcal{O}(2^{(g_j-1)(r-M)}), \qquad \mathcal{P}[1, 1, \ldots |r] = 1 + \mathcal{O}(2^{r-M})$$

Since k is finite and $\Phi(1, 1, 1, \ldots) = 1$, the total contribution to $\log(\mu_k)$ from those terms where $M - r \to \infty$ as $M \to \infty$ is negligible, since for $1 \ll R \ll M$ we may write

$$\begin{aligned}&\sum_{r=0}^{R-1} \log\Big[\sum_{(g_1,g_2,\ldots)} \delta_{k,g_1+g_2+\ldots}\Phi(g_1, g_2, \ldots)\mathcal{P}[g_1, g_2, \ldots |r]\Big] \\ &\quad = \sum_{r=0}^{R-1} \log\Big[1 + \mathcal{O}(2^{r-M})\Big] = \mathcal{O}(2^{R-M})\end{aligned} \tag{101}$$

Hence in (100) we need only those terms where $M - r$ is finite. These terms represent contributions where *virtually all* past components of the history strings at the times $\{\ell_1, \ldots, \ell_k\}$ were identical, which should indeed constrain the possible histories at the times $\{\ell_1, \ldots, \ell_k\}$ most, and indeed gives the largest history coincidence rates. We consequently switch our conditioning label from the number r of previously identical components to the number $M - r$ of unconstrained components, and write

$$\mathcal{P}[g_1, g_2, \ldots |r] = \mathcal{Q}[g_1, g_2, \ldots |M - r]$$

* For example: (k) denotes the subset of all combinations $\{\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_k\}$ where $\boldsymbol{\lambda}_1 = \ldots = \boldsymbol{\lambda}_k$, $(2, 1, 1, \ldots)$ is the subset of all $\{\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_k\}$ where precisely two history strings are identical, and all others are distinct.

and find (100) converting into the simpler form

$$\lim_{M\to\infty} \log(\mu_k) = \sum_{r\geq 1} \log\Big[\sum_{(g_1,g_2,\ldots)} \delta_{k,g_1+g_2+\ldots}\Phi(g_1,g_2,\ldots)\mathcal{Q}[g_1,g_2,\ldots|r]\Big] \tag{102}$$

We are left with the task to calculate the likelihood $\mathcal{Q}[g_1,g_2,\ldots|r]$ of finding at the k distinct times $\{\ell_1,\ldots,\ell_k\}$ of our process the histories $\{\boldsymbol{\lambda}_1,\ldots,\boldsymbol{\lambda}_k\}$ to be identical in prescribed multiples of $(g_1,g_2,\ldots)$, given that the bits of the k history vectors were identical during all but r of the most recent iterations.

At this stage we benefit from having to consider only values of r in (102) which are finite (compared to M, which is sent to infinity). For each value r of the number of 'free' components, there will be only 2^r possible history strings $\boldsymbol{\lambda}$ available to be allocated to the k times $\{\ell_1,\ldots,\ell_k\}$. In principle one would have to worry about the probabilities to be assigned to each of the 2^r options. However, we know for the full M-component history strings that their probabilities scale as $\pi_{\boldsymbol{\lambda}} = f_{\boldsymbol{\lambda}}p^{-1}$ with $f_{\boldsymbol{\lambda}} = \mathcal{O}(1)$, so the effective probabilities of individual components of $\boldsymbol{\lambda}\in\{-1,1\}$ must scale as

$$\pi_{\lambda_i} = \mathcal{O}(\pi_{\boldsymbol{\lambda}}^{1/M}) = \mathcal{O}(\frac{1}{2}f_{\boldsymbol{\lambda}}^{1/M}) = \frac{1}{2}[1+\mathcal{O}(M^{-1})]$$

From this we deduce that for finite r we may take all 2^r allowed history strings to have equal probabilities. This turns the evaluation of $\mathcal{Q}[g_1,g_2,\ldots|r]$ into a solvable combinatorial problem. Each of k elements is given randomly one of 2^r colours (where each colour has probability 2^{-r}), and $\mathcal{Q}[g_1,g_2,\ldots|r]$ represents the likelihood of finding identical colour sets of sizes $(g_1,g_2,\ldots)$. Let us abbreviate $R=2^r$, and write the r-th term in (102) as $\log(H_r)$. Now, using $2^{-r(g_1+\ldots g_R)} = 2^{-rk} = R^{-k}$ we may simply write♯

$$\lim_{M\to\infty} \log(\mu_k) = \sum_{r\geq 1} \log H_r \tag{103}$$

$$\begin{aligned} H_r =& \sum_{(g_1,g_2,\ldots)} \Phi(g_1,g_2,\ldots)\mathcal{Q}[g_1,g_2,\ldots|r] \\ =& \sum_{g_1=0}^{k}\sum_{g_2=0}^{k-g_1}\sum_{g_3=0}^{k-g_1-g_2}\cdots\sum_{g_R=0}^{k-g_1-\ldots-g_{R-1}} \Phi(g_1,g_2,\ldots)\ \delta_{k,\sum_i g_i} \\ & \times R^{-k}\binom{k}{g_1}\binom{k-g_1}{g_2}\binom{k-g_1-g_2}{g_3}\cdots\binom{k-g_1-\ldots-g_{R-1}}{g_R} \end{aligned} \tag{104}$$

6.3. Expansion of sign-coincidence probabilities

Having simplified the conditional distribution $\mathcal{Q}[g_1,g_2,\ldots|r]$ of history coincidences, we turn to $\Phi(g_1,g_2,\ldots)$ as given by (99). If we restrict ourselves to an expansion of (99) in powers of the (random) bid biases $\overline{A}_{\boldsymbol{\lambda}}$ in which we retain only the leading terms, our

♯ One easily confirms that our expression for H_r is properly normalized. Upon choosing $\Phi(g_1,g_2,\ldots)=1$ one can perform the summations iteratively, starting from g_R and descending down to g_1, which leads exactly to the factor R^k to combine with the R^{-k} present.

problem simplifies further to the point where we can obtain a fully explicit expression for the moments μ_k. In Appendix B we derive the following compact relations:

$$\Phi(1,1,1,\ldots) = 1 \tag{105}$$

$$\Phi(g_1,g_2,\ldots) = e^{\frac{1}{2}\Omega\sum_{j\geq 1} g_j(g_j-1)-\frac{1}{4}\Omega^2\sum_j g_j(g_j-1)(2g_j-3)+\mathcal{O}(\Omega^3)} \tag{106}$$

$$\Omega = \sum_{\boldsymbol{\lambda}} \pi_{\boldsymbol{\lambda}}\ \mathrm{Erf}^2\Big[\frac{(1-\zeta)\overline{A}_{\boldsymbol{\lambda}}}{\sqrt{2}\sqrt{\zeta^2\kappa^2+(1-\zeta)^2\sigma^2_{\boldsymbol{\lambda}}}}\Big] \tag{107}$$

The results (105,106) imply that, rather than knowing the full probability distribution $\mathcal{P}[g_1,g_2,\ldots|r]$ in (100), we only need the (conditional) statistics of a modest number of relatively simple monomials. Expanding the exponential in (106) up to the relevant orders, and using $\sum_j g_j = k$ (which is always true inside (104)) produces

$$\begin{aligned}\Phi(g_1,g_2,\ldots) &= 1+\frac{1}{2}\Omega\Big[\sum_{j\geq 1} g_j^2 - k\Big] \\ &+ \frac{1}{4}\Omega^2\Big[\frac{1}{2}\sum_{ij\geq 1} g_i^2 g_j^2 - 2\sum_{j\geq 1} g_j^3 - (k-5)\sum_{j\geq 1} g_j^2 + \frac{1}{2}k^2 - 3k\Big] + \mathcal{O}(\Omega^3)\end{aligned} \tag{108}$$

Since the combinatorial averaging process of (104) in this particular representation involves a measure which is invariant under permutations of the numbers $\{g_1,g_2,\ldots\}$, the average of (108) is identical to that of the following simpler function (with $R=2^r$):

$$\begin{aligned}\Phi_{\mathrm{eff}}(g_1,g_2,\ldots) &= 1+\frac{1}{2}\Omega(Rg_1^2-k) \\ &+\frac{1}{8}\Omega^2\Big[Rg_1^4+R(R-1)g_1^2g_2^2-4Rg_1^3-2(k-5)Rg_1^2+k^2-6k\Big]+\mathcal{O}(\Omega^3)\end{aligned} \tag{109}$$

Instead of having to use full combinatorial measure (104), we can therefore extract all the relevant information from the (joint) marginal distribution for the pair (g_1,g_2) only. Inserting (109) into (104) gives us

$$\begin{aligned}H_r = 1+\frac{1}{2}\Omega\Big[RG^{k,R}_{2,0}-k\Big]+\frac{1}{8}\Omega^2\Big[RG^{k,R}_{4,0}+R(R-1)G^{k,R}_{2,2} \\ -4RG^{k,R}_{3,0}-2(k-5)RG^{k,R}_{2,0}+k^2-6k\Big]+\mathcal{O}(\Omega^3)\end{aligned} \tag{110}$$

with

$$\begin{aligned}G^{k,R}_{a,b} &= \sum_{g_1=0}^{k}\sum_{g_2=0}^{k-g_1}\sum_{g_3=0}^{k-g_1-g_2}\cdots\sum_{g_R=0}^{k-g_1-\ldots-g_{R-1}} g_1^a g_2^b\ \delta_{k,\sum_i g_i} \\ &\quad\times R^{-k}\binom{k}{g_1}\binom{k-g_1}{g_2}\binom{k-g_1-g_2}{g_3}\cdots\binom{k-g_1-\ldots-g_{R-1}}{g_R} \\ &= R^{-k}\sum_{g_1=0}^{k}\sum_{g_2=0}^{k-g_1}\binom{k}{g_1}\binom{k-g_1}{g_2}(R-2)^{k-g_1-g_2}g_1^a g_2^b\end{aligned} \tag{111}$$

Those combinatorial factors $G^{k,R}_{a,b}$ which we need in order to evaluate (110) are calculated in Appendix C. They are found to be

$$G^{k,R}_{2,0} = \frac{k}{R}+\frac{k(k-1)}{R^2}$$

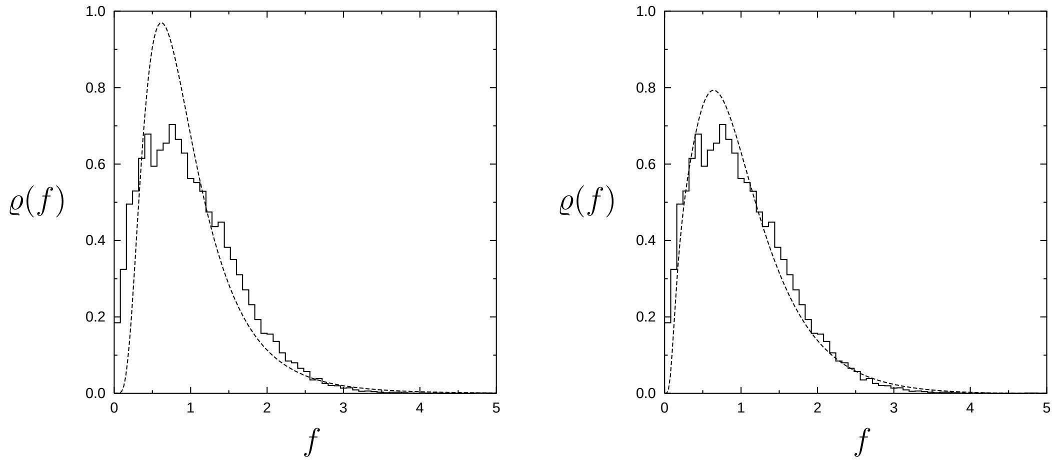

Figure 2. Test of the predicted history frequency distributions (116) (left picture, based on expansion of the moments μ_k up to first order in the width, $\mu_k = e^{\frac{1}{2}\Omega k(k-1)}$) and (118) (right picture, based on expansion up to second order, $\mu_k = e^{\frac{1}{2}\Omega k(k-1)-\frac{1}{12}\Omega^2 k(k-1)(2k-3)}$), together with the data of Fig. 1 as measured in simulations for $\alpha = 2.0$ and $N = 8193$. In both cases the second moment which parametrizes (116) and (118) was taken from the data: $\mu_2 \approx 1.380$.

$$G^{k,R}_{3,0} = \frac{k}{R} + \frac{3k(k-1)}{R^2} + \frac{k(k-1)(k-2)}{R^3}$$
$$G^{k,R}_{4,0} = \frac{k}{R} + \frac{7k(k-1)}{R^2} + \frac{6k(k-1)(k-2)}{R^3} + \frac{k(k-1)(k-2)(k-3)}{R^4}$$
$$G^{k,R}_{2,2} = \frac{k(k-1)}{R^2} + \frac{2k(k-1)(k-2)}{R^3} + \frac{k(k-1)(k-2)(k-3)}{R^4}$$

Insertion of these factors into (110), followed by restoration of the short-hand $R = 2^r$, gives us the fully explicit expression

$$H_r = 1 + \frac{1}{2}\Omega k(k-1)2^{-r} + \frac{1}{8}\Omega^2 k(k-1)(k-2)(k-3)4^{-r} + \mathcal{O}(\Omega^3) \quad (112)$$

We can now write explicit formulae for the moments of the relative history frequencies, and hence also for the distribution $\varrho(f)$ itself, in the form an expansion in a parameter Ω which controls the width of this distribution.

6.4. *Resulting prediction for* $\varrho(f)$

The result (112), together with the earlier relation (103) and the geometric series leads us finally to the desired expression for the moments μ_k:

$$\lim_{M\to\infty} \log(\mu_k) = \frac{1}{2}\Omega k(k-1) - \frac{1}{12}\Omega^2 k(k-1)(2k-3) + \mathcal{O}(\Omega^3) \quad (113)$$

We see that this general formula obeys $\mu_0 = \mu_1 = 1$, as it should, and that

$$\lim_{M\to\infty} \mu_2 = e^{\Omega - \frac{1}{6}\Omega^2 + \mathcal{O}(\Omega^3)} \quad (114)$$

Insertion into our earlier expression (89) for $\varrho(f)$ leads in the limit $M \to \infty$ to a formula in which, at least up the relevant orders in Ω, the insertion of a Gaussian integral allows

us to carry out the summation over moments explicitly:

$$\begin{aligned}
\varrho(f) &= \int \frac{d\omega}{2\pi} e^{i\omega f} \sum_{k\geq 0} \frac{(-i\omega)^k}{k!} e^{\frac{1}{2}\Omega k(k-1)-\frac{1}{12}\Omega^2 k(k-1)(2k-3)+\mathcal{O}(\Omega^3)} \\
&= \int Dz \int \frac{d\omega}{2\pi} e^{i\omega f} \sum_{k\geq 0} \frac{(-i\omega)^k}{k!} \Big[1 - \frac{1}{6}\sqrt{\Omega}\frac{d^3}{dz^3} + \ldots\Big] e^{zk\sqrt{\Omega+\frac{5}{6}\Omega^2}-\frac{1}{2}k(\Omega+\frac{1}{2}\Omega^2)} \\
&= \int Dz \Big[1 + \frac{1}{6}\sqrt{\Omega}(3z - z^3) + \ldots\Big]\, \delta\Big[f - e^{z\sqrt{\Omega+\frac{5}{6}\Omega^2}-\frac{1}{2}(\Omega+\frac{1}{2}\Omega^2)}\Big] \qquad (115)
\end{aligned}$$

We may use (114) to express Ω in terms of μ_2, turning our expansion of the moments μ_k into an expansion in powers of $\log(\mu_2)$. Depending on whether we wish to take our expansion only to order $\mathcal{O}(\log(\mu_2))$, or also to $\mathcal{O}(\log^2(\mu_2))$, we obtain

$$\text{to } \mathcal{O}(\log(\mu_2)): \qquad \varrho(f) = \frac{e^{-\frac{1}{2}z^2(f)}}{f\sqrt{2\pi \log(\mu_2)}} \qquad (116)$$

$$z(f) = \frac{\log(f) + \frac{1}{2}\log(\mu_2)}{\sqrt{\log(\mu_2)}} \qquad (117)$$

$$\text{to } \mathcal{O}(\log^2(\mu_2)): \qquad \varrho(f) = \frac{e^{-\frac{1}{2}z^2(f)}\Big[1 + \frac{1}{6}\sqrt{\log(\mu_2)}(3z(f) - z^3(f))\Big]}{f\sqrt{2\pi[\log(\mu_2) + \log^2(\mu_2)]}} \qquad (118)$$

$$z(f) = \frac{\log(f) + \frac{1}{2}[\log(\mu_2) + \frac{2}{3}\log^2(\mu_2)]}{\sqrt{\log(\mu_2) + \log^2(\mu_2)}} \qquad (119)$$

The two statements (116) and (118) are indeed found to constitute increasingly accurate predictions for the actual distribution of the relative history frequencies, see e.g. Fig. 2. We have thus been able to explain the origin and the characteristics of the observed history frequency statistics. However, both formulae are expansions for small Ω. Should (118) be applied to values of Ω which are not small, one has to be careful in dealing with large values of f, where $\varrho(f)$ could become negative (this would have been prevented by the higher orders in Ω). The implication is that in the Gaussian integral (115) one must in practice either introduce a cut-off $z_c = \mathcal{O}(\Omega^{-1/6})$, or exponentiate the factor $[1 + \frac{1}{6}\sqrt{\log(\mu_2)}(3z(f) - z^3(f))]$.

6.5. The width of $\varrho(f)$

What remains in order to round off our analysis of the distribution of relative history frequencies is to calculate the width parameter Ω in (113) self-consistently from our equations. According to our theory, Ω is given by (107), i.e. by

$$\Omega = \sum_{\boldsymbol{\lambda}} \pi_{\boldsymbol{\lambda}}\ \mathrm{Erf}^2\Big[\frac{(1-\zeta)\overline{A}_{\boldsymbol{\lambda}}}{\sqrt{2}\sqrt{\zeta^2\kappa^2 + (1-\zeta)^2\sigma^2_{\boldsymbol{\lambda}}}}\Big] \qquad (120)$$

The quantities $\overline{A}_{\boldsymbol{\lambda}}$ and $\sigma^2_{\boldsymbol{\lambda}} = \overline{A^2}_{\boldsymbol{\lambda}} - \overline{A}_{\boldsymbol{\lambda}}$ describe the statistics of those bids which correspond to times with a prescribed history string $\boldsymbol{\lambda}$. We know from (64) that these are Gaussian variables, which implies that $\overline{A}_{\boldsymbol{\lambda}}$ and $\sigma^2_{\boldsymbol{\lambda}}$ are all we need to know. Since we

restrict ourselves to non-anomalous TTI states, we can write both as long-time averages:

$$\overline{A}_{\boldsymbol{\lambda}} = \pi_{\boldsymbol{\lambda}}^{-1} \lim_{L\to\infty} L^{-1} \sum_{\ell=1}^{L} \delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ell,A,Z)} A(\ell) \tag{121}$$

$$\overline{A^2}_{\boldsymbol{\lambda}} = \pi_{\boldsymbol{\lambda}}^{-1} \lim_{L\to\infty} L^{-1} \sum_{\ell=1}^{L} \delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ell,A,Z)} A^2(\ell) \tag{122}$$

We can work out the average $\overline{A}_{\boldsymbol{\lambda}}$, using (64) and time-translation invariance, and subsequently define the new time variables $s_i = \ell_i - \ell_{i+1}$ (for $i < r$) and $s_r = \ell_r$ (so that $\ell_j = s_j + s_{j+1} + \ldots + s_r$). This results in

$$\begin{aligned}
\overline{A}_{\boldsymbol{\lambda}} &= \lim_{L\to\infty} \frac{1}{Lp\pi_{\boldsymbol{\lambda}}} \sum_{\ell_0=1}^{L} \sum_{r\geq 0} (-\delta_N)^r \sum_{\ell_1\ldots\ell_r} G(\ell_0-\ell_1)\ldots G(\ell_{r-1}-\ell_r) \\
&\qquad \times \Big[\prod_{i=0}^{r} p\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ell_i,A,Z)}\Big]\phi_{\ell_r} \\
&= \frac{1}{p\pi_{\boldsymbol{\lambda}}} \sum_{r\geq 0} (-\delta_N)^r \sum_{s_0\ldots s_{r-1}} G(s_0)\ldots G(s_{r-1}) \\
&\qquad \lim_{L\to\infty} \frac{1}{L} \sum_{s_r=0}^{L-\sum_{i<r} s_i} \Big[\prod_{i=0}^{r} p\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(s_i+\ldots+s_r,A,Z)}\Big]\phi_{s_r}
\end{aligned} \tag{123}$$

Given our ansatz of short history correlation times, in the sense of (79), and given $\chi = \sum_{\ell>0} G(\ell) < \infty$ (so $G(\ell)$ must decay sufficiently fast), we find this expression simplifying to

$$\overline{A}_{\boldsymbol{\lambda}} = \sum_{r\geq 0} (-\chi p\pi_{\boldsymbol{\lambda}})^r \lim_{L\to\infty} \frac{1}{\pi_{\boldsymbol{\lambda}} L} \sum_{s=0}^{L} \delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(s,A,Z)} \phi_s = \frac{\overline{\phi}_{\boldsymbol{\lambda}}}{1+\chi p\pi_{\boldsymbol{\lambda}}} \tag{124}$$

In a similar manner we find

$$\begin{aligned}
\overline{A^2}_{\boldsymbol{\lambda}} &= \lim_{L\to\infty} \frac{1}{\pi_{\boldsymbol{\lambda}} L} \sum_{\ell_0\ell_0'=0}^{L} \delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ell_0,A,Z)} A(\ell_0) A(\ell_0') \delta_{\ell_0\ell_0'} \\
&= \lim_{L\to\infty} \frac{1}{\pi_{\boldsymbol{\lambda}} L} \sum_{\ell_0\ell_0'=0}^{L} \sum_{r,r'\geq 0} (-\delta_N)^{r+r'} \sum_{\ell_1\ldots\ell_r} G(\ell_0-\ell_1)\ldots G(\ell_{r-1}-\ell_r) \\
&\quad \times \sum_{\ell_1'\ldots\ell_r'} G(\ell_0'-\ell_1')\ldots G(\ell_{r'-1}'-\ell_{r'}')\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ell_0,A,Z)} \\
&\quad \times \Big[\prod_{i=1}^{r} p\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ell_i,A,Z)}\Big]\Big[\prod_{i=1}^{r'} p\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ell_i',A,Z)}\Big]\delta_{\ell_0\ell_0'}\phi_{\ell_r}\phi_{\ell_{r'}'} \\
&= \frac{1}{p\pi_{\boldsymbol{\lambda}}} \sum_{r,r'\geq 0} (-\delta_N)^{r+r'} \sum_{s_0\ldots s_{r-1}} G(s_0)\ldots G(s_{r-1}) \sum_{s_0'\ldots s_{r'-1}'} G(s_0')\ldots G(s_{r'-1}') \\
&\quad \times \lim_{L\to\infty} \frac{1}{L} \sum_{s_r=0}^{L-\sum_{i<r} s_i} \sum_{s_{r'}'=0}^{L-\sum_{i<r'} s_i'} p\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(s_0+\ldots+s_r,A,Z)} \\
&\quad \times \Big[\prod_{i=1}^{r} p\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(s_i+\ldots+s_r,A,Z)}\Big]\Big[\prod_{i=1}^{r'} p\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(s_i'+\ldots+s_{r'}',A,Z)}\Big]\delta_{\sum_i s_i,\sum_i s_i'}\phi_{s_r}\phi_{s_{r'}'}
\end{aligned} \tag{125}$$

Again we use $\sum_\ell G(\ell)<\infty$ to justify that in the summations over s_r and $s'_{r'}$ the upper limit can safely be replaced by L. Thus:

$$
\begin{aligned}
\overline{A^2}_{\boldsymbol{\lambda}} = \frac{1}{p\pi_{\boldsymbol{\lambda}}} \sum_{r,r'\geq 0} (-\delta_N)^{r+r'} \sum_{s_0\ldots s_{r-1}} G(s_0)\ldots G(s_{r-1}) \sum_{s'_0\ldots s'_{r'-1}} G(s'_0)\ldots G(s'_{r'-1}) \\
\times \lim_{L\to\infty}\frac{1}{L}\sum_{s_r=0}^{L} p\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\sum_j s_j,A,Z)} \Big[\prod_{i=1}^{r} p\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\sum_{j\geq i} s_j,A,Z)}\Big] \\
\times \Big[\prod_{i=1}^{r'} p\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\sum_j s_j-\sum_{j<i}s'_j,A,Z)}\Big]\phi_{s_r}\phi_{\sum_j s_j-\sum_{j<r'}s'_j}
\end{aligned}
\tag{126}
$$

The present calculation is similar to that of the volatility matrix in the fake history online MG [10] (the quantity $\sigma^2_{\boldsymbol{\lambda}}=\overline{A^2}_{\boldsymbol{\lambda}}-\overline{A}^2_{\boldsymbol{\lambda}}$ can be regarded as a conditional volatility, where the condition is that in collecting our statistics we are to restrict ourselves to those times where the observed history strings take the value $\boldsymbol{\lambda}$), so also here we have to worry about pairwise time coincidences. Each such coincidence effectively removes one constraint of the type $\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ldots,A,Z)}$, since the latter will be met automatically. The remaining terms will occur in extensive summations, so that we may replace each 'unpaired' occurrence of a factor $\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ldots,A,Z)}$, except for those with the same argument as one of the Gaussian variables ϕ, by its time average $\pi_{\boldsymbol{\lambda}}$. In practice this implies the replacement

$$
\begin{aligned}
\Big[\prod_{i=1}^{r-1} p\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\sum_{j\geq i}s_j,A,Z)}\Big]\Big[\prod_{i=1}^{r'-1} p\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\sum_j s_j-\sum_{j<i}s'_j,A,Z)}\Big] \to \\
(p\pi_{\boldsymbol{\lambda}})^{r+r'-2}\pi_{\boldsymbol{\lambda}}^{-\sum_{i=1}^{r-1}\sum_{j=1}^{r'-1}\delta_{\sum_{\ell\geq i}s_\ell,\sum_\ell s_\ell-\sum_{\ell<j}s'_\ell}} \\
=(p\pi_{\boldsymbol{\lambda}})^{r+r'-2}\prod_{i=1}^{r-1}\prod_{j=1}^{r'-1}\Big[1+\frac{1-\pi_{\boldsymbol{\lambda}}}{p\pi_{\boldsymbol{\lambda}}}\frac{\tilde{\eta}}{2\delta_N}\delta_{\sum_{\ell<i}s_\ell,\sum_{\ell<j}s'_\ell}\Big]
\end{aligned}
\tag{127}
$$

and therefore

$$
\begin{aligned}
\overline{A^2}_{\boldsymbol{\lambda}} = \sum_{r,r'\geq 0}(-\delta_N)^{r+r'}\sum_{s_0\ldots s_{r-1}} G(s_0)\ldots G(s_{r-1})\sum_{s'_0\ldots s'_{r'-1}} G(s'_0)\ldots G(s'_{r'-1}) \\
\times (p\pi_{\boldsymbol{\lambda}})^{r+r'}\prod_{i=1}^{r-1}\prod_{j=1}^{r'-1}\Big[1+\frac{1-\pi_{\boldsymbol{\lambda}}}{p\pi_{\boldsymbol{\lambda}}}\frac{\tilde{\eta}}{2\delta_N}\delta_{\sum_{\ell<i}s_\ell,\sum_{\ell<j}s'_\ell}\Big] \\
\times \sum_k \delta_{k,\sum_{j<r}s_j-\sum_{j<r'}s'_j}\lim_{L\to\infty}\frac{1}{\pi^2_{\boldsymbol{\lambda}}L}\sum_{s=0}^{L}\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(s,A,Z)}\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(s+k,A,Z)}\phi_s\phi_{s+k}
\end{aligned}
\tag{128}
$$

As in the calculation of the volatility in [10], lacking as yet a method to deal with all the complicated terms generated by the factor proportional to the learning rate $\tilde{\eta}$, we have to restrict ourselves in practice to approximations. As in [10] we first remove the most tricky terms by putting $\tilde{\eta}\to 0$. This gives

$$
\begin{aligned}
\overline{A^2}_{\boldsymbol{\lambda}} = \sum_{r,r'\geq 0}(-\delta_N p\pi_{\boldsymbol{\lambda}})^{r+r'}\sum_{s_0\ldots s_{r-1}} G(s_0)\ldots G(s_{r-1})\sum_{s'_0\ldots s'_{r'-1}} G(s'_0)\ldots G(s'_{r'-1}) \\
\times \sum_k \delta_{k,\sum_{j<r}s_j-\sum_{j<r'}s'_j}\lim_{L\to\infty}\frac{1}{\pi^2_{\boldsymbol{\lambda}}L}\sum_{s=0}^{L}\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(s,A,Z)}\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(s+k,A,Z)}\phi_s\phi_{s+k}
\end{aligned}
\tag{129}
$$

We then assume that the limit $L \to \infty$ in the last line converts the associated sample average into a full average over the statistics of the Gaussian fields ϕ_ℓ given by (62), i.e.

$$\begin{aligned}
&\lim_{L\to\infty} \frac{1}{\pi^2_{\boldsymbol{\lambda}} L} \sum_{s=0}^{L} \delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(s,A,Z)} \delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(s+k,A,Z)} \phi_s \phi_{s+k} \;\to \\
&\qquad \lim_{L\to\infty} \frac{1}{\pi^2_{\boldsymbol{\lambda}} L} \sum_{s=0}^{L} \delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(s,A,Z)} \delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(s+k,A,Z)} \langle \phi_s \phi_{s+k} \rangle_{\phi|A,Z} = \frac{1}{2}[1 + C(k)]
\end{aligned}$$

Separating the correlation function into a persistent and a non-persistent term, $C(k) = c + \tilde{C}(k)$, and returning to the earlier notation with time differences inside the kernels G, results in the history-conditioned equivalent of the volatility approximation in [10]:

$$\begin{aligned}
\overline{A^2}_{\boldsymbol{\lambda}} &= \sum_{r,r'\geq 0} (-\delta_N p \pi_{\boldsymbol{\lambda}})^{r+r'} \sum_{s_0 \ldots s_{r-1}} G(s_0) \ldots G(s_{r-1}) \sum_{s'_0 \ldots s'_{r'-1}} G(s'_0) \ldots G(s'_{r'-1}) \\
&\quad \times \frac{1}{2}\Big[1 + c + \tilde{C}(\sum_{j<r} s_j - \sum_{j<r'} s'_j)\Big] \\
&= \frac{1+c}{2(1+\chi p \pi_{\boldsymbol{\lambda}})^2} + \frac{1}{2}\int ds ds'\; (\mathbb{1} + p\pi_{\boldsymbol{\lambda}} G)^{-1} \tilde{C}(s-s') (\mathbb{1} + p\pi_{\boldsymbol{\lambda}} G)^{-1} \qquad (130)
\end{aligned}$$

where $\mathbb{1}(x,y) = \delta(x-y)$. In order to get to the present stage we have averaged the ϕ-dependent terms inside $\overline{A^2}_{\boldsymbol{\lambda}}$ over the Gaussian measure $\langle \ldots \rangle_{\phi|A,Z}$. Consistency demands that in working out $\sigma^2_{\boldsymbol{\lambda}} = \overline{A^2}_{\boldsymbol{\lambda}} - \overline{A}^2_{\boldsymbol{\lambda}}$ we do the same with the term $\overline{A}^2_{\boldsymbol{\lambda}}$, where $\overline{A}_{\boldsymbol{\lambda}}$ is given by (124), so our approximation for the history-conditioned volatility becomes

$$\begin{aligned}
\boldsymbol{\sigma}^2_{\boldsymbol{\lambda}} &= \overline{A^2}_{\boldsymbol{\lambda}} - \frac{\langle \overline{\phi}^2_{\boldsymbol{\lambda}} \rangle}{(1+\chi p \pi_{\boldsymbol{\lambda}})^2} \\
&= \overline{A^2}_{\boldsymbol{\lambda}} - \lim_{L\to\infty} \frac{1}{(L\pi_{\boldsymbol{\lambda}})^2} \sum_{\ell\ell'=1}^{L} \delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ell,A,Z)} \delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ell',A,Z)} \frac{1 + C(\ell-\ell')}{2(1+\chi p \pi_{\boldsymbol{\lambda}})^2} \\
&= \frac{1}{2}\int ds ds'\; (\mathbb{1} + p\pi_{\boldsymbol{\lambda}} G)^{-1}(s) \tilde{C}(s-s') (\mathbb{1} + p\pi_{\boldsymbol{\lambda}} G)^{-1}(s') \qquad (131)
\end{aligned}$$

Our final step again follows [10]. We assume that the non-persistent correlations $\tilde{C}(t)$ decay vary fast, away from the value $\tilde{C}(0) = 1 - c$, so that in the expansion of (131) in powers of G we retain only the zero-th term:

$$\boldsymbol{\sigma}^2_{\boldsymbol{\lambda}} = \frac{1}{2}(1-c) \qquad (132)$$

We may now return to expression (120) and insert our approximations (124) and (132):

$$\begin{aligned}
\Omega &= \lim_{p\to\infty} \sum_{\boldsymbol{\lambda}} \pi_{\boldsymbol{\lambda}}\; \mathrm{Erf}^2 \left[\frac{(1-\zeta)\overline{\phi}_{\boldsymbol{\lambda}}}{\sqrt{2}(1+\chi p \pi_{\boldsymbol{\lambda}}) \sqrt{\zeta^2\kappa^2 + \frac{1}{2}(1-\zeta)^2(1-c)}} \right] \\
&= \int df d\phi\; \varrho(f,\phi) f\; \mathrm{Erf}^2 \left[\frac{(1-\zeta)\phi}{\sqrt{2}(1+\chi f) \sqrt{\zeta^2\kappa^2 + \frac{1}{2}(1-\zeta)^2(1-c)}} \right] \qquad (133)
\end{aligned}$$

with

$$\varrho(f,\phi) = \lim_{p\to\infty} \frac{1}{p} \sum_{\boldsymbol{\lambda}} \delta[f - p\pi_{\boldsymbol{\lambda}}] \delta[\phi - \overline{\phi}_{\boldsymbol{\lambda}}] \qquad (134)$$

We know the $\overline{\phi}_{\boldsymbol{\lambda}}$ to be Gaussian variables, with $\langle\overline{\phi}_{\boldsymbol{\lambda}}\rangle = 0$ and $\langle\overline{\phi}^2_{\boldsymbol{\lambda}}\rangle = \frac{1}{2}(1+c)$ (see the above derivation of $\sigma^2_{\boldsymbol{\lambda}}$ where this was shown and used). Hence, upon making our final simplifying assumption that in the relevant orders of our calculation the correlations between the history frequencies $\pi_{\boldsymbol{\lambda}}$ and the Gaussian fields $\overline{\phi}_{\boldsymbol{\lambda}}$ are irrelevant, we obtain

$$\varrho(f,\phi) = \varrho(f)\ \frac{e^{-\phi^2/(1+c)}}{\sqrt{\pi(1+c)}} \tag{135}$$

and hence (133) simplifies to

$$\Omega = \int_0^\infty df\ \varrho(f) f \int Dx\ \mathrm{Erf}^2\left[\frac{x(1-\zeta)\sqrt{1+c}}{2(1+\chi f)\sqrt{\zeta^2\kappa^2+\frac{1}{2}(1-\zeta)^2(1-c)}}\right] \tag{136}$$

Using the integral $\int Dx\ \mathrm{Erf}^2(Ax) = \frac{4}{\pi}\arctan[\sqrt{1+4A^2}] - 1$, in combination with the identity $\int df\ \varrho(f) f = 1$, our approximate expression for the parameter Ω thus becomes

$$\Omega = \frac{4}{\pi}\int_0^\infty df\ \varrho(f) f\ \ \arctan\left[1+\frac{(1-\zeta)^2(1+c)}{(1+\chi f)^2[\zeta^2\kappa^2+\frac{1}{2}(1-\zeta)^2(1-c)]}\right]^{\frac{1}{2}} - 1 \tag{137}$$

In the limit of strictly fake history we recover from (137) the value $\lim_{\zeta\to 1}\Omega = (4/\pi)\arctan[1] - 1 = 0$, as it should. For MGs with strictly true market history, on the other hand, expression (137) simplifies to

$$\lim_{\zeta\to 0}\Omega = \frac{4}{\pi}\int_0^\infty df\ \varrho(f) f\ \ \arctan\left[1+\frac{2(1+c)}{(1+\chi f)^2(1-c)}\right]^{\frac{1}{2}} - 1 \tag{138}$$

In accordance with earlier observations in simulations [15] we also see that, as the system approaches the phase transition when α is lowered from within the ergodic regime, the increase of the susceptibility χ automatically reduces the width parameter Ω, until it vanishes completely at the critical point.

7. Closed theory for persistent observables in the ergodic regime

We have now obtained a closed theory for the time-translation invariant states of our MG, albeit in approximation. It consists of the equations (84,85,86,87) for the persistent order parameters, combined with expressions (116,118) for the shape and (114,137) for the width of the relative history frequency distribution $\varrho(f)$. This theory predicts correctly (i) that the phase transition point $\alpha_c(T)$ of the MG with history is identical to that of the model with fake memory, (ii) that at the transition point the relative history frequency distribution reduces to $\varrho(f) = \delta[f-1]$ (with at that point also the order parameters all becoming independent of whether we have true or fake history), and (iii) the shape of the relative history frequency distribution. In the limit $\alpha\to\infty$ the theory also reproduces the correct order parameter values $\chi = \phi = c = 0$, for any value of ζ. For $\zeta = 0$ (strictly true memory) it predicts $\lim_{\alpha\to\infty}\Omega = \frac{1}{3}$ and hence $\lim_{\alpha\to\infty}\mu_2 \approx 1.37$.

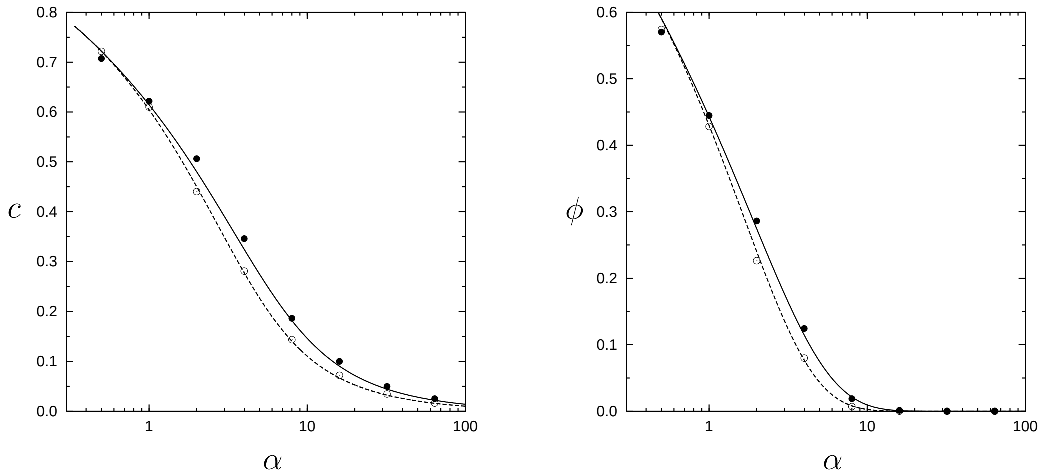

Figure 3. Left: the predicted persistent correlations c together with simulation data in the non-ergodic regime, for the on-line MG with strictly true history (i.e. $\zeta = 0$; the solid line gives the theoretical prediction, full circles the experimental data) and for the on-line MG with strictly fake memory (i.e. $\zeta = 1$; the dashed line gives the theoretical prediction, open circles the experimental data). In both cases decision noise was absent. Right: the corresponding predicted fraction ϕ of frozen agents, under the same experimental conditions and with the same meaning of lines and markers.

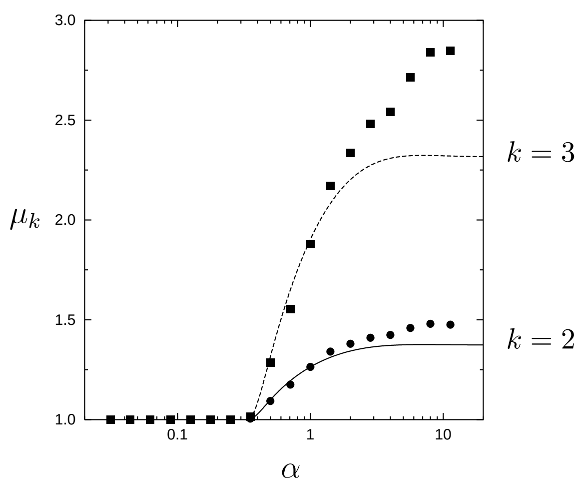

Figure 4. The moments $\mu_2 = \int df\ \varrho(f) f^2$ and $\mu_3 = \int df\ \varrho(f) f^3$ of the distribution of relative history frequencies for the MG with strictly true history and absent decision noise (i.e. $\zeta = T = 0$), as predicted by the theory (solid and dashed lines), compared to the moments as measured in numerical simulations (markers, with circles indicating μ_2 and squares indicating μ_3). Note that $\mu_0 = \mu_1 = 1$ (by definition).

Let us finally reduce our closed equations to a more compact form, for the simplest nontrivial case of the MG with strictly true market history (i.e. $\zeta = 0$) and without decision noise (i.e. $\sigma[\infty] = 1$). Here we have

$$u = \frac{\sqrt{\alpha}\chi_R}{S_0\sqrt{2}} \qquad \chi = \frac{1-\phi}{\alpha\chi_R} \qquad \phi = 1 - \mathrm{Erf}[u] \tag{139}$$

$$c \;= 1-\mathrm{Erf}[u]+\frac{1}{2u^2}\mathrm{Erf}[u]-\frac{1}{u\sqrt{\pi}}e^{-u^2} \tag{140}$$

$$\chi_{_R} = \int Dz\Big[1+\frac{1}{6}\sqrt{\Omega}(3z-z^3)\Big]\Big[e^{-z\sqrt{\Omega+\frac{5}{6}\Omega^2}+\frac{1}{2}(\Omega+\frac{1}{2}\Omega^2)}+\chi\Big]^{-1} \tag{141}$$

$$S_0^2 = (1+c)\int Dz\Big[1+\frac{1}{6}\sqrt{\Omega}(3z-z^3)\Big]\Big[e^{-z\sqrt{\Omega+\frac{5}{6}\Omega^2}+\frac{1}{2}(\Omega+\frac{1}{2}\Omega^2)}+\chi\Big]^{-2} \tag{142}$$

$$\begin{aligned}\Omega \;=& \int Dz\Big[1+\frac{1}{6}\sqrt{\Omega}(3z-z^3)\Big]e^{z\sqrt{\Omega+\frac{5}{6}\Omega^2}-\frac{1}{2}(\Omega+\frac{1}{2}\Omega^2)}\\ &\times\left\{\frac{4}{\pi}\arctan\Big[1+\frac{2(1+c)}{(1-c)\Big[1+\chi e^{z\sqrt{\Omega+\frac{5}{6}\Omega^2}-\frac{1}{2}(\Omega+\frac{1}{2}\Omega^2)}\Big]^2}\Big]^{\frac{1}{2}}-1\right\}\end{aligned} \tag{143}$$

Upon using (140) to write c as a function of u, i.e. $c=c(u)$ with $c(u)$ denoting the right-hand side of (140), and upon eliminating the quantities ϕ and S_0, we find ourselves with a closed set of equations for the trio $\{u,\chi,\Omega\}$:

$$u = \frac{\mathrm{Erf}[u]}{\chi\sqrt{2\alpha(1+c)}}\left\{\int Dz\frac{1+\frac{1}{6}\sqrt{\Omega}(3z-z^3)}{\Big[e^{-z\sqrt{\Omega+\frac{5}{6}\Omega^2}+\frac{1}{2}(\Omega+\frac{1}{2}\Omega^2)}+\chi\Big]^2}\right\}^{-\frac{1}{2}} \tag{144}$$

$$\chi = \frac{\mathrm{Erf}[u]}{\alpha}\left\{\int Dz\frac{1+\frac{1}{6}\sqrt{\Omega}(3z-z^3)}{e^{-z\sqrt{\Omega+\frac{5}{6}\Omega^2}+\frac{1}{2}(\Omega+\frac{1}{2}\Omega^2)}+\chi}\right\}^{-1} \tag{145}$$

$$\begin{aligned}\Omega =& \int Dz\Big[1+\frac{1}{6}\sqrt{\Omega}(3z-z^3)\Big]e^{z\sqrt{\Omega+\frac{5}{6}\Omega^2}-\frac{1}{2}(\Omega+\frac{1}{2}\Omega^2)}\\ &\times\left\{\frac{4}{\pi}\arctan\Big[1+\frac{2[1+c(u)]}{[1-c(u)]\Big[1+\chi e^{z\sqrt{\Omega+\frac{5}{6}\Omega^2}-\frac{1}{2}(\Omega+\frac{1}{2}\Omega^2)}\Big]^2}\Big]^{\frac{1}{2}}-1\right\}\end{aligned} \tag{146}$$

Solving these three coupled equations numerically, followed by comparison with simulation data, shows a surprising level of agreement, in spite of the expansions and assumptions which have been used to derive (144,145,146). Figure 3 shows the performance of the theory in describing the on-line MG with strictly true market history (i.e. $\zeta=0$), together with similar data for the on-line fake history MG (i.e. $\zeta=1$), for comparison††. In all these simulations $N=8193$. Calculation of the first two non-trivial moments μ_k of the distribution of relative history frequencies, see e.g. figure 4 (where in the simulations $\alpha N^2=2^{28}$), shows that for small values of the width of $\varrho(f)$ (i.e. μ_2 close to one, which is true close to and below the critical point) the predictions of the theory are excellent, but that the performance of equations (144,145,146) deteriorates for larger values of μ_2. This is obvious, since these equations result effectively from an expansion for small values of μ_2-1. Taking this expansion to higher orders should lead to systematic improvement, but will be non-trivial.

†† Below the critical point, where $\chi=\infty$ throughout, equation (137) predicts that $\Omega=0$. This implies that $\varrho(f)=\delta[f-1]$ for $\alpha<\alpha_c(T)$, and that below the critical point the differences between true and fake history (if any) are confined to dynamical phenomena or to states without time-translation invariance. This confirms earlier observations in numerical simulations [15], where it was found that the persistent order parameters in MGs with and without history were identical in the low α regime.

8. Discussion

We have developed a mathematical procedure for the derivation of exact dynamical solutions for Minority Games with *real* market histories, using the generating functional analysis techniques of [17]. So far these techniques had only been developed for (and applied successfully to) the less realistic but mathematically simpler MG versions with fake market histories, restricting theoretical progress to those particular game versions only. We have shown how the technical difficulties associated with the non-Markovian character of the microscopic laws induced by having real histories can be dealt with, and found (in the infinite system size limit) exact and closed macroscopic laws from which to solve the canonical dynamic order parameters for the standard (on-line) MG with true market history. Here these laws turn out to be formulated in terms of *two* effective equations (rather than a single equation, as for models with fake histories): one for an effective agent, and one for an effective overall market bid. In the second part of this paper we have constructed solutions for these effective equations, focusing mostly on the usual persistent observables of the MG in time-translation invariant states (persistent correlations and the fraction of frozen agents) and on the calculation from first principles of the distribution of history frequencies. These objects are calculated in the form of an expansion in powers of the width of the history frequency distribution, of which the first few terms are derived in explicit form. The final theory was shown to give accurate predictions for the persistent observables and for the shape of the history frequency distribution. It gives precise predictions for the width in the region where this width remains relatively small (which is inevitable in view of the expansion used).

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Appendix A. Recovering the fake history limit

It helpful for our understanding of the $N \to \infty$ limit in (59,60) to first return to the simplest case where we know what the outcome should be, being $\zeta = 1$, i.e. fake history strings of the inconsistent type (4). This is the model which was solved in [10]. In doing so we *en passant* re-confirm the correctness of the assumed scaling $\delta_N = \tilde{\eta}/2p$.

For $\zeta = 1$ we see in (15) and (17) that both $\boldsymbol{\lambda}(\ldots)$ and $\overline{W}[\ldots]$ lose their dependence on the path $\{A\}$, and reduce to

$$\boldsymbol{\lambda}(\ell, Z) = \begin{pmatrix} \mathrm{sgn}[Z(\ell,1)] \\ \vdots \\ \mathrm{sgn}[Z(\ell,M)] \end{pmatrix} \qquad \overline{W}[\ell,\ell';Z] = \delta_{\boldsymbol{\lambda}(\ell,Z),\boldsymbol{\lambda}(\ell',Z)} \tag{A.1}$$

The role of the Gaussian variables $\{Z\}$ has thereby been reduced to determining the statistics of the symmetric random matrix $\mathcal{B}$ with entries $\mathcal{B}_{\ell\ell'} = \overline{W}[\ell,\ell';Z]$:

$$\mathcal{P}[\mathcal{B}] = \Big\langle \prod_{\ell,\ell'} \delta\Big[\mathcal{B}_{\ell\ell'} - \prod_{\lambda=1}^{\log_2(p)} \theta[Z(\ell,\lambda)Z(\ell',\lambda)]\Big]\Big\rangle_{\{Z\}} \tag{A.2}$$

with $p = 2^M = \alpha N$. The two relevant properties of these matrices are relatively easily derived, and are found to be the following. For any cyclic combination of r-th moments (with $r > 0$), where $s_1 > s_2 > \ldots > s_r$ and $r > 1$ (no summations) one has

$$\langle \mathcal{B}_{s_1 s_2}\mathcal{B}_{s_2,s_3}\ldots\mathcal{B}_{s_r,s_1}\rangle_{\mathcal{B}} = p^{1-r} \tag{A.3}$$

The second type of average one needs involves two time-ordered strings of matrix elements (of lengths r and r', respectively) connected by two further matrix elements, where $s_0 > s_1 > \ldots > s_r$ and $s'_0 > s'_1 > \ldots > s'_{r'}$:

$$\begin{aligned}\langle [\mathcal{B}_{s_0 s_1}\mathcal{B}_{s_1 s_2}\ldots\mathcal{B}_{s_{r-1}s_r}]\mathcal{B}_{s_r s'_{r'}}[\mathcal{B}_{s'_0 s'_1}\mathcal{B}_{s'_1 s'_2}\ldots\mathcal{B}_{s'_{r'-1}s'_{r'}}]\mathcal{B}_{s_0 s'_0}\rangle \\ = p^{\sum_{i=0}^{r}\sum_{j=0}^{r'}\delta_{s_i s'_j} - r - r' - 1}\end{aligned} \tag{A.4}$$

The partial decoupling of the paths $\{A\}$ and $\{Z\}$ implies that our expressions for the kernels R and Σ simplify to

$$R(t,t') = \lim_{\delta_N \to 0} \frac{\delta}{\delta A_e(t')} \left.\Big\langle \mathcal{B}_{\ell\ell'}\,\langle A(\ell)\rangle_{\{A|\mathcal{B}\}}\Big\rangle_{\mathcal{B}}\right|_{\ell=t/\delta_N,\ell'=t'/\delta_N} \tag{A.5}$$

$$\Sigma(t,t') = \tilde{\eta}\lim_{\delta_N \to 0}\frac{1}{\delta_N}\left.\Big\langle \mathcal{B}_{\ell\ell'}\,\langle A(\ell)A(\ell')\rangle_{\{A|\mathcal{B}\}}\Big\rangle_{\mathcal{B}}\right|_{\ell=t/\delta_N,\ell'=t'/\delta_N} \tag{A.6}$$

Since the bid evolution process (49) is now linear in $\{A\}$, and involves only $\{A\}$-independent zero-average Gaussian fields ϕ_ℓ, where $\langle \phi_\ell \phi_{\ell'}\rangle_{\{\phi|\mathcal{B}\}} = \frac{1}{2}\mathcal{B}_{\ell\ell'}[1 + C(\ell,\ell')]$, it is easily solved for any given realization of the random matrix $\mathcal{B}$:

$$A(\ell) = A_e(\ell) + \phi_\ell + \sum_{r>0}(-\frac{\tilde{\eta}}{2})^r \sum_{k<\ell}[(G\mathcal{B})^r]_{\ell k}\;[A_e(k) + \phi_k] \tag{A.7}$$

in which $G\mathcal{B}$ denotes the matrix with entries $(G\mathcal{B})_{\ell\ell'} = G(\ell,\ell')\mathcal{B}_{\ell\ell'}$ (i.e. involving component multiplication rather than matrix multiplication). To make a comparison

with the results of [10] we must remove the external bid perturbations $A_e(\ell)$ after they have served to generate the response function R.

We can evaluate (A.5) using only expression (A.7), the causality of the response function, and formula (A.3). These give, with $\delta/\delta A_e(\ell) = {\delta_N}^{-1}\partial/\partial A(\ell)$:

$$
\begin{aligned}
R(\ell,\ell') &= \lim_{A_e\to 0}\lim_{\delta_N\to 0} \frac{\partial}{\partial A_e(\ell')}\frac{1}{\delta_N}\int d\mathcal{B}\ \mathcal{P}[\mathcal{B}]\ \mathcal{B}_{\ell\ell'}\ \langle A(\ell)\rangle_{\{\phi|\mathcal{B}\}} \\
&= \lim_{\delta_N\to 0}\frac{1}{\delta_N}\sum_{r\geq 0}(-\frac{\tilde{\eta}}{2})^r\int d\mathcal{B}\ \mathcal{P}[\mathcal{B}]\ \mathcal{B}_{\ell\ell'}[(G\mathcal{B})^r]_{\ell\ell'} \\
&= \lim_{\delta_N\to 0}\frac{1}{\delta_N}\Big\{\delta_{\ell\ell'} - \delta_N G(\ell,\ell') \\
&\qquad + \sum_{r>1}(-\delta_N)^r\sum_{s_2>s_3>\ldots>s_r} G(\ell,s_2)G(s_2,s_3)\ldots G(s_r,\ell')\Big\}
\end{aligned}
\tag{A.8}
$$

We observe in (A.8), in view of $\delta_{\ell\ell'}\to\delta_N\delta(t-t')$ in the limit $N\to\infty$, that the canonical scaling of time (modulo $\mathcal{O}(1)$ factors) is indeed $\delta_N = \tilde{\eta}/2p$. We then find exactly the expression in [10] for the on-line 'fake history' MG:

$$
R(t,t') = \delta(t-t') + \sum_{r>0}(-1)^r G^r(t,t') = [\mathbf{1}+G]^{-1}(t,t') \tag{A.9}
$$

Had we chosen an alternative scaling with N of δ_N, we would have found either the trivial result $R=0$, or an ill-defined expression.

Next we turn to expression (60) for the effective agent's noise covariances, with $A_e = 0$. The equivalence of the present expression and that in [10] will be more transparent upon renaming $(\ell,\ell')\to(s_0,s_0')$ and $D(k,k') = 1 + C(k,k')$:

$$
\begin{aligned}
\Sigma(s_0,s_0') &= \lim_{\delta_N\to 0}\frac{\tilde{\eta}}{\delta_N}\int d\mathcal{B}\ \mathcal{P}[\mathcal{B}]\ \mathcal{B}_{s_0 s_0'}\ \langle A(s_0)A(s_0')\rangle_{\{\phi|\mathcal{B}\}} \\
&= \lim_{\delta_N\to 0}\frac{\tilde{\eta}}{2\delta_N}\sum_{r,r'\geq 0}(-\frac{\tilde{\eta}}{2})^{r+r'}\sum_{s_1\ldots s_r}\sum_{s_1'\ldots s_{r'}'} D(s_r,s_{r'}') \\
&\quad\times\ G(s_0,s_1)\ldots G(s_{r-1},s_r)\ G(s_0',s_1')\ldots G(s_{r'-1}',s_{r'}') \\
&\quad\times\ \langle(\mathcal{B}_{s_0 s_1}\ldots\mathcal{B}_{s_{r-1}s_r})\mathcal{B}_{s_r s_{r'}'}(\mathcal{B}_{s_0' s_1'}\ldots\mathcal{B}_{s_{r-1}' s_r'})\mathcal{B}_{s_0 s_0'}\rangle_{\mathcal{B}}
\end{aligned}
\tag{A.10}
$$

with the proviso that when $r=0$ we must interpret the sums as $\sum_{s_1\ldots s_2}\to 1$, $G(s_0,s_1)\ldots G(s_{r-1},s_r)\to 1$ and $\mathcal{B}_{s_0 s_1}\ldots\mathcal{B}_{s_{r-1}s_r}\to 1$ (and similarly when $r'=0$). Since the kernel $\Sigma(s_0,s_0')$ is symmetric, we may without loss of generality choose $s_0'\geq s_0$. Dependent on the whether any or both of the indices (r,r') are zero, we have to evaluate the following averages (with the short-hand $\overline{\delta}_{ij} = 1-\delta_{ij}$):

- $r=r'=0$: here the average of the last line in (A.10) reduces to
$$
\langle\ldots\rangle_{\mathcal{B}} = \langle\mathcal{B}_{s_0 s_0'}^2\rangle = \langle\mathcal{B}_{s_0 s_0'}\rangle = \delta_{s_0 s_0'} + \frac{1}{p}\overline{\delta}_{s_0 s_0'} \tag{A.11}
$$
- $r'=0,\ r>0$: here the average in (A.10) reduces to two terms (representing the cases $s_0=s_0'$ versus $s_0<s_0'$), which are both of the form (A.3),
$$
\langle\ldots\rangle_{\mathcal{B}} = \langle(\mathcal{B}_{s_0 s_1}\ldots\mathcal{B}_{s_{r-1}s_r})\mathcal{B}_{s_r s_0'}\mathcal{B}_{s_0 s_0'}\rangle = p^{-r}\delta_{s_0 s_0'} + p^{-r-1}\overline{\delta}_{s_0 s_0'} \tag{A.12}
$$
where we used $\mathcal{B}_{kk}=1$, for any k. The case $r=0$, $r'>0$ is clearly equivalent.

- $r, r' > 0$: now the relevant average reduces to that of (A.4),

$$
\begin{aligned}
\langle \ldots \rangle_{\mathcal{B}} &= \langle [\mathcal{B}_{s_0 s_1} \ldots \mathcal{B}_{s_{r-1} s_r}] \mathcal{B}_{s_r s'_{r'}} [\mathcal{B}_{s'_0 s'_1} \ldots \mathcal{B}_{s'_{r'-1} s'_{r'}}] \mathcal{B}_{s_0 s'_0} \rangle \\
&= p^{\sum_{i=0}^{r} \sum_{j=0}^{r'} \delta_{s_i s'_j} - r - r' - 1}
\end{aligned}
\tag{A.13}
$$

Expression (A.13) reduces to those derived for the cases where r or r' is zero (or both), so it is true for any (r, r'). We may thus insert (A.13) into (A.10), and obtain:

$$
\begin{aligned}
\Sigma(s_0, s'_0) = &\lim_{\delta_N \to 0} \sum_{r, r' \geq 0} (-\delta_N)^{r+r'} \sum_{s_1 \ldots s_r} \sum_{s'_1 \ldots s'_{r'}} D(s_r, s'_{r'}) \prod_{i=0}^{r} \prod_{j=0}^{r'} \Big[1 + (p-1)\delta_{s_i s'_j}\Big] \\
&\times\; G(s_0, s_1) \ldots G(s_{r-1}, s_r)\; G(s'_0, s'_1) \ldots G(s'_{r'-1}, s'_{r'}) \\
= &\lim_{N \to \infty} \sum_{r, r' \geq 0} (-\delta_N)^{r+r'} \sum_{s_1 \ldots s_r} \sum_{s'_1 \ldots s'_{r'}} D(s_r, s'_{r'}) \\
&\times \prod_{i=0}^{r} \prod_{j=0}^{r'} \Big[1 + \frac{\tilde{\eta}}{2\delta_N} \delta_{s_i s'_j} [1 - \mathcal{O}(\delta_N)]\Big] \\
&\times\; G(s_0, s_1) \ldots G(s_{r-1}, s_r)\; G(s'_0, s'_1) \ldots G(s'_{r'-1}, s'_{r'}) \\
= &\sum_{r, r' \geq 0} (-1)^{r+r'} \int_0^{\infty} ds_1 \ldots ds_r ds'_1 \ldots ds'_{r'} \prod_{i=0}^{r} \prod_{j=0}^{r'} \Big[1 + \frac{1}{2} \tilde{\eta} \delta[s_i - s'_j]\Big] \\
&\times\; G(s_0, s_1) \ldots G(s_{r-1}, s_r)\; G(s'_0, s'_1) \ldots G(s'_{r'-1}, s'_{r'}) \\
&\times\; [1 + C(s_r, s'_{r'})]
\end{aligned}
\tag{A.14}
$$

Also (A.14) is identical to the corresponding expression in [10], as it should.

Appendix B. Expansion of bid sign recurrence probabilities

Here we derive the expansion (106) of the function $\Phi(g_1, g_2, \ldots)$ as defined in (99). We abbreviate

$$
E_{\boldsymbol{\lambda}} = \mathrm{Erf}\Big[\frac{(1-\zeta)\overline{A}_{\boldsymbol{\lambda}}}{\sqrt{2}\sqrt{\zeta^2 \kappa^2 + (1-\zeta)^2 \sigma^2_{\boldsymbol{\lambda}}}}\Big]
\tag{B.1}
$$

with $\sum_{\boldsymbol{\lambda}} \pi_{\boldsymbol{\lambda}} E^r_{\boldsymbol{\lambda}} = \langle E^r \rangle$. These short-hands allow us to compactify (99) to

$$
\begin{aligned}
\Phi(g_1, g_2, \ldots) &= \frac{1}{2} \prod_{j \geq 1} \Big[\sum_{\boldsymbol{\lambda}} \pi_{\boldsymbol{\lambda}} (1 + E_{\boldsymbol{\lambda}})^{g_j}\Big] + \frac{1}{2} \prod_{j \geq 1} \Big[\sum_{\boldsymbol{\lambda}} \pi_{\boldsymbol{\lambda}} (1 - E_{\boldsymbol{\lambda}})^{g_j}\Big] \\
&= \frac{1}{2} \prod_{j \geq 1} \Big[\sum_{n=0}^{g_j} \binom{g_j}{n} \langle E^n \rangle\Big] + \frac{1}{2} \prod_{j \geq 1} \Big[\sum_{n=0}^{g_j} \binom{g_j}{n} (-1)^n \langle E^n \rangle\Big] \\
&= \sum_{n_1=0}^{g_1} \sum_{n_2=0}^{g_2} \cdots \binom{g_1}{n_1} \binom{g_2}{n_2} \cdots \frac{1}{2} [1 + (-1)^{n_1 + n_2 + \ldots}] \langle E^{n_1} \rangle \langle E^{n_2} \rangle \ldots
\end{aligned}
\tag{B.2}
$$

Since the overall average bid in the MG is equally likely to be positive than negative, and since (B.1) tells us that $\mathrm{sgn}[E_{\boldsymbol{\lambda}}] = \mathrm{sgn}[\overline{A}_{\boldsymbol{\lambda}}]$, the moments $\langle E^r \rangle$ for even values of r

will have to be zero. From this it follows that

$$\Phi(g_1,g_2,\ldots)=\sum_{0\leq n_1\leq \frac{1}{2}g_1}\sum_{0\leq n_2\leq \frac{1}{2}g_2}\cdots\binom{g_1}{2n_1}\binom{g_2}{2n_2}\cdots\langle E^{2n_1}\rangle\langle E^{2n_2}\rangle\cdots$$
$$=\prod_{j\geq 1}\left[1+\sum_{1\leq n\leq g_j/2}\binom{g_j}{2n}\langle E^{2n}\rangle\right]$$

so

$$\log\Phi(g_1,g_2,\ldots)=\sum_{j\geq 1}\log\Big[1+\sum_{1\leq n\leq g_j/2}\binom{g_j}{2n}\langle E^{2n}\rangle\Big] \tag{B.3}$$

Equation (B.3) tells us, firstly, that

$$\Phi(1,1,1,\ldots)=1 \tag{B.4}$$

For arbitrary history coincidence numbers $(g_1,g_2,\ldots)$, not necessarily all equal to one, we may expand (B.3) in the moments $\langle E^r\rangle$:

$$\log\Phi(g_1,g_2,\ldots)=\sum_{j\geq 1}\log\Big[1+\frac{1}{2}g_j(g_j-1)\langle E^2\rangle+\frac{1}{24}g_j(g_j-1)(g_j-2)(g_j-3)\langle E^4\rangle$$
$$+\mathcal{O}(\langle E^6\rangle)\Big]$$
$$=\frac{1}{2}\sum_{j\geq 1}g_j(g_j-1)\left\{\langle E^2\rangle+\frac{1}{12}\Big[(g_j-2)(g_j-3)\langle E^4\rangle-3g_j(g_j-1)\langle E^2\rangle^2\Big]\right\}+\mathcal{O}(\langle E^6\rangle)$$

Finally, in leading order in E we may regard the variables $E_{\boldsymbol{\lambda}}$ as proportional to $\overline{A}_{\boldsymbol{\lambda}}$, and therefore as distributed in a Gaussian manner. This implies (since $\langle E\rangle=0$) that in leading order we have $\langle E^4\rangle=3\langle E^2\rangle$. Hence

$$\log\Phi(g_1,g_2,\ldots)=\frac{1}{2}\langle E^2\rangle\sum_{j\geq 1}g_j(g_j-1)-\frac{1}{4}\langle E^2\rangle^2\sum_{j\geq 1}g_j(g_j-1)(2g_j-3)$$
$$+\mathcal{O}(\langle E^6\rangle) \tag{B.5}$$

Appendix C. Combinatorics in history frequency moments

In this appendix we calculate the combinatorial factors $G^{k,R}_{a,b}$ as defined in (111). They can be obtained by differentiation of a simple generating function:

$$G^{k,R}_{a,b}=R^{-k}\sum_{g_1=0}^{k}\sum_{g_2=0}^{k-g_1}\binom{k}{g_1}\binom{k-g_1}{g_2}(R-2)^{k-g_1-g_2}g_1^a g_2^b$$
$$=R^{-k}\lim_{x,y\to 1}(x\frac{d}{dx})^a(y\frac{d}{dy})^b(R-2+x+y)^k \tag{C.1}$$

In particular:

$$G^{k,R}_{2,0}=kR^{-1}+k(k-1)R^{-2} \tag{C.2}$$
$$G^{k,R}_{3,0}=kR^{-1}+3k(k-1)R^{-2}+k(k-1)(k-2)R^{-3} \tag{C.3}$$
$$G^{k,R}_{4,0}=kR^{-1}+7k(k-1)R^{-2}+6k(k-1)(k-2)R^{-3}+k(k-1)(k-2)(k-3)R^{-4} \tag{C.4}$$
$$G^{k,R}_{2,2}=k(k-1)R^{-2}+2k(k-1)(k-2)R^{-3}+k(k-1)(k-2)(k-3)R^{-4} \tag{C.5}$$